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Ultimate Boundedness Stability and Controllability of Hereditary Systems

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Ultimate Boundedness Stability and Controllability of Hereditary Systems

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I. Preface

Very many realistic models assume that the state of the system under study depend both on the past values of the state as well as on the past values of the derivative of the state. Such systems are governed by neutral functional differential equations (NFDE). When the system is dependent on the past values of the state but is independent of the past values of the derivative the system's dynamics may be described by a retarded functional differential equation. Often the principle of causality prevails and the future state of the system is independent of the past and is determined solely by the present. Such systems can often be assumed to be described by ordinary differential equations. One very powerful tool used to study the asymptotic behavior of ordinary differential equations is the generalized energy functions of the models. Yoshizawa [1,2] utilized them to give necessary and sufficient conditions for the uniform boundedness and uniform ultimate boundedness of ordinary and delay equations. A recent attempt was made by Lopes [3] to use such functions to obtain a sufficient condition for all solutions of and NFDE to be uniformly bounded and uniformly ultimately bounded. This effort was not quite complete and differs somewhat from the ordinary and delay cases. The first chapter of this report defines a class of NFDE for which it is possible to develop a theory of boundedness and ultimate boundedness of solutions using Yoshizawa type functions. As a by-product the existence of periodic solutions for such NFDE's is proved. Numerous specific applications are also given.

In chapter III, we apply the theory developed in chapter II, to prove the uniform boundedness and uniform ultimate boundedness of a neutral functional differential equation of Lurie type. Inspired by a similar treatment by Burton [14] we also explore the boundedness problem for wide class of general ordinary and delay equations. In chapter IV the problem of Lurie (which already has had an extensive history [16,17]) is posed for NFDE. Using a theorem of Cruz and Hale [5], sufficient conditions are obtained for absolute stability for the controlled system if it is assumed that the uncontrolled plant equation is uniformly asymptotically stable. Both the direct and indirect control cases are treated.

The next two chapters examine some fundamental questions concerning the controllability of systems governed by retarded equations and NFDE when it is assumed that the controls are constrained to lie on some compact convex subsets of the Euclidean space E^m. For linear functional differential equations of retarded type in which the power available is unlimited and controls are unconstrained, Banks, Jacobs and Langenhop [35] have derived necessary and sufficient conditions for controllability in the Sobolev space $W_2^{(1)}$. Jacobs and Langenhop [23] have obtained necessary and sufficient conditions for the controllability in $W_2^{(1)}$ for linear autonomous differential difference equations of neutral type with unconstrained controls. In chapter V we consider the nonlinear neutral functional differential inclusion

$$\frac{d}{dt} D(t,x_t) \in R(t,x_t) \tag{1}$$

where D is a continuous operator on I×C, linear in x_t, indeed of the form (4) in chapter V, with kernel $D(t,\cdot) = \{0\}$, and atomic at 0, and R is nonempty, closed and convex. Here $I \equiv [t_0,t_1]$, and $C = C([-h,0],E^n)$. We use Fan fixed point theorem [22] to prove the existence of a solution of the inclusion (1) which satisfies two point boundary values $x_{t_0} = \phi_0$, $x_{t_1} = \phi_1$, where ϕ_0,ϕ_1 belong to C. We next apply this existence result to study the exact function space controllability of the neutral functional differential system

$$\frac{d}{dt} D(t,x_t) = f(t,x_t,u), \quad u(t) \in \Omega(t,x_t) . \tag{2}$$

We present sufficient conditions on f and Ω which imply exact controllability between two fixed functions in C.

In chapter VI, we return to systems described by delay equations and consider both the Euclidean and the function space controllability of the control system

$$\dot{x}(t) = L(t,x_t) + k(t,u), \quad t \geq t_0 , \tag{3}$$

$$x(0) \equiv 0 \qquad t \in [t_0-h,t_0] ,$$

when the available control power is limited and the controls have values restricted to compact and convex subsets of E^m. We use a geometric growth condition to characterize both types of controllability This extends analogous results for ordinary differential systems [39]-[41] and yields conditions under which perturbed nonlinear delay controllable systems are controllable.

The concluding chapter indicates further areas of research.

II. Ultimate Boundedness of Solutions of Some Nonlinear Neutral Differential Equations

1. Introduction. In this chapter we study the uniform-boundedness and uniform ultimate boundedness of solutions of certain types of neutral functional differential equations. For delay systems, Yoshizawa [1] treated such problems by extending the ideas of Liapunov: he used Liapunov functionals whose properties are similar to those he utilized in his study of ordinary differential equations [2]. In [3], Lopes combined Liapunov functionals and Razumikhin techniques to prove ultimate boundedness. Aside from the fact that the applications of his theory involves the verification of more conditions than are used in cases [1] and [2], the results in [3] are not quite complete. His conditions are sufficient but not necessary. In this paper Razumikhin techniques are avoided; sufficient and necessary conditions are given for uniform boundedness and uniform ultimate boundedness. in terms of Liapunov functionals alone. This reduces the number of conditions to be verified.

An important aspect of our contribution is the treatment of several non-trivial linear and nonlinear examples. Explicit Liapunov functionals are constructed, and the existence of periodic solutions deduced by applying a result of Hale and Lopes [4] on the existence of periodic solutions of compact dissipative systems. Unlike the applications in [2] the use of Liapunov functionals does not put some restrictions on the delay.

2. Notation, Definitions and Preliminaries. Let $h \geq 0$ be a given real number, E^n, an n-dimensional linear vector space with norm $|\cdot|$ and $C = C([-h,0],E^n)$ the space of continuous functions mapping $[-h,0]$ into E^n with $\|\phi\| = \sup_{-h\leq\theta\leq 0} |\phi(\theta)|$

for $\phi \in C$. Let t_0 be a real number and f,g are continuous functions taking $[t_0,\infty)\times C \to E^n$. Assume that g is linear in ϕ and that there exists an n×n-matrix function $\mu(t,\theta)$, $t \in [t_0,\infty)$, $\theta \in [-h,0]$ which is of bounded variation in θ and there exists a scalar function $\ell(\theta)$ continuous and non-decreasing for $s \in [0,h]$, $\ell(\theta) = 0$ such that

$$g(t,\phi) = \int_{-h}^{0} [d_\theta \mu(t,\theta)]\phi(\theta)$$

$$\left|\int_{-h}^{0} [d_\theta \mu(t,\theta)]\phi(\theta)\right| \leq \ell(h) \sup_{-h\leq\theta\leq 0} |\phi(\theta)|$$

for all $t \in [t_0,\infty)$, $\phi \in C$.

Define a functional differential operator

$$D(\cdot): [t_0,\infty)\times C \to E^n$$

by

$$D(t)\phi = \phi(0) - g(t,\phi),\ t \in [t_0,\infty),\ \phi \in C.$$

We shall study the functional differential equation of neutral type given by

$$\frac{d}{dt} D(t)x_t = f(t,x_t),\ t > t_0 \tag{1}$$

$$x_{t_0} = \phi$$

where $x_t \in C$ is defined by

$$x_t(\theta) = x(t+\theta),\ -h \leq \theta \leq 0.$$

We shall assume that $f(t,\phi)$ is locally Lipschitzian in ϕ and takes bounded sets into bounded sets to ensure the existence, uniqueness and continuous dependence of the solution $x(t_0,\phi)$ of (1) on the initial data. Here $x(t_0,\phi)$ is the solution with $x_{t_0}(t_0,\phi) = \phi$ and its value in E^n at time t will be denoted by $x(t;\ t_0,\phi)$. Observe that the initial value problem (1) is equivalent to the integral equation

$$D(t)x_t = D(t_0)\phi + \int_{t_0}^{t} f(t,x_t)ds,\ t \geq t_0. \tag{2}$$

In Cruz and Hale [5] the concept of a uniformly stable operator was introduced and was shown to imply the following.

Definition 2.1. The operator D is uniformly stable if there are constants $\beta,\alpha > 0$ such that the solution $x(t_0,\phi)$ of the "difference equation"

$$D(t)x_t = 0,\ t \geq t_0 \tag{3}$$

$$x_{t_0} = \phi,\ D(t_0)\phi = 0$$

satisfies $\|x_t(t_0,\phi)\| \leq \beta e^{-\alpha(t-t_0)}\|\phi\|,\ t \geq t_0$.

Definition 2.2. We say that the solutions of (1) are uniform bounded if for each $\alpha > 0$ there exists $\beta(\alpha)$ such that if $\|\phi\| \leq \alpha$ then $\|x_t(t_0,\phi)\| \leq \beta(\alpha)$. We say that the solutions of (1) are uniform ultimate bounded for bound B if there exists $B > 0$ and for each $\alpha > 0$ there exists $T(\alpha)$ such that if $\|\phi\| \leq \alpha$ we have $\|x_t(t_0,\phi)\| \leq B$ for all $t \geq t_0 + T(\alpha)$. If $V: [t_0,\infty)\times C \to E$ is continuous we define the "derivative" $\dot{V}(t,\phi)$ along solution (2) as

$$\dot{V}(t,\phi) = \dot{V}_{(2)}(t,\phi) = \overline{\lim_{h\to 0^+}} \frac{1}{h}[V(t,x_{t+h}(t,\phi) - V(t,\phi)].$$

The following two lemmas are crucial in our investigation.

Lemma 2.1. (Cruz-Hale [5]). Suppose the A_k, $k=1,2,\ldots,N$ are $n\times n$ constant matrices τ_k, $0 \leq \tau_k \leq h$ are real numbers such that the ratio τ_j/τ_k are rational if $N > 0$. If

$$D(\phi) = \phi(0) - \sum_{k=1}^{N} A_k\phi(-\tau_k) \tag{4}$$

and all roots of the equation

$$\det[I - \sum_{k=1}^{N} A_k\rho^{-\tau_k}] = 0$$

have moduli less than 1, then D is uniformly stable.

Lemma 2.2. (Cruz-Hale [5]). If $D(t)$ is uniformly stable, then there exists positive constants a,b,c,d such that for any $h \in C([t_0,\infty),E^n)$ the solution $x(t_0,\phi,h)$ of the equation

$$D(t)x_t = h(t),\ t \geq t_0,\ x_{t_0} = \phi \tag{5}$$

satisfies

$$\|x_t(t_0,\phi,h)\| \leq e^{-a(t-t_0)}(b\|\phi\| + c \sup_{t_0\leq u\leq t} |h(u)|) + d \sup_{\sigma\leq u\leq t} |h(u)| \tag{6}$$

for all $t \geq t_0$. Furthermore, the constants a,b,c,d can be chosen so that for any $s \in [t_0,\infty)$

$$\|x_t(t_0,\phi,h)\| \leq e^{-a(t-s)}(b\|\phi\| + c \sup_{\sigma\leq u\leq t} |h(u)|) + d \sup_{s\leq u\leq t} |h(u)|. \tag{7}$$

Definition 2.3. Let $\frac{d}{dt} D(t)x_t = f(t,x_t)$ with D and f ω-periodic. We say it defines a compact dissipative process if there is a bounded set $B \subset C$ such that for any compact set H there is a $T_0(H)$ such that

$$x(t,t_0,H) \subseteq B \text{ for all } t \geq t_0 + T_0(H).$$

Clearly a uniform ultimate bounded process is compact dissipative.

Proposition 2.1. [4]. If the ω-periodic NFDE $(\cdot)$ with uniformly stable D operator defines a compact dissipative process, then it has an ω-periodic solution.

3. Ultimate Boundedness of Solutions of (2). In this section we shall obtain a sufficient condition for boundedness by means of a Liapunov functional. Throughout what follows the set S of ϕ is the set of $\phi \in C$ such that $\|\phi\| \geq R$ where R may be a large positive constant.

Theorem 3.1. Suppose there exists a continuous functional $V: [0,\infty)\times S$ such that

(i) $u(|D(t)\phi|) \le V(t,\phi) \le v(\|\phi\|)$

where $u(r)$ is continuous increasing positive for $r > R$ and $u(r) \to \infty$ as $r \to \infty$, and $v(r)$ is continuous and increasing;

(ii) $\dot{V}(t,\phi) \le - w(|D(t)\phi|)$,

where $w(r)$ is continuous and positive for $r > R$.

If $D(t)$ is uniformly stable then the solutions of (2) are uniformly bounded and uniform ultimate bounded.

Proof. Suppose the constants a,b,c,d are defined as in Lemma 2.2. For any $\alpha > 0$ $(\alpha > R)$ choose $\beta(\alpha)$ so that

$$b\alpha < \frac{\beta}{2}, \quad v(\alpha) < u(\beta/2(c+d)).$$

If $\phi \in C$ and $\|\phi\| \le \alpha$, suppose that at some t_1 we have $\|x_{t_1}(t_0,\phi)\| = \beta$. Since $\|x_t(t_0,\phi)\|$ is continuous in t and $\|x_{t_0}(t_0,\phi)\| \le \alpha$, there exist t_2, t_3, $t_0 \le t_2 \le t_3 \le t_1$, such that $\|x_{t_1}(t_0,\phi)\| = \alpha$, $|x(t_3; t_0,\phi_0)| = \beta$, and for $t_2 < t < t_3$ we have

$$\alpha < \|x_t(t_0,\phi)\| < \beta.$$

Also for $t_2 \le t \le t_3 \le t_1$, $\|x_t(t_0,\phi)\| \ge \alpha > R$. Hence $x_t(t_0,\phi) \in S$. Consider the function $V(\cdot, x_t(t_0,\phi))$. Condition (ii) implies that $V(t,x_t(t_0,\phi))$ is non-increasing and condition (i) implies that

$$u(|D(t)x_t(t_0,\phi)|) \le V(t,x_t(t_0,\phi)) \le V(t_0,\phi) \le v(\alpha) < u(\beta/2(c+d)).$$

Consequently

$$|D(t)x_t(t_0,\phi)| < \beta/2(c+d)$$

for all $t \in [t_2,t_1]$. Since $D(t)$ is stable we deduce from Lemma 2 that

$$\|x_t(t_0,\phi)\| \le b\|\phi\| + (c+d)\beta/2(c+d) \le b\alpha + \beta/2 < \beta.$$

That is $\|x_t(t_0,\phi)\| < \beta$ for all $t \in [t_2,t_1]$. But $\|x_{t_1}[t_0,\phi)\| = \beta$. This is a contradiction. Hence if $\|\phi\| \le \alpha$ we have $\|x_t(t_0,\phi)\| < \beta(\alpha)$ for all $t \ge t_0$, which proves uniform boundedness.

For a fixed $\alpha_1 > \alpha$ where $\ell = \lim[\alpha_1 \ \frac{\alpha_1}{4d}] > R$ there exists a $\beta_1(\alpha_1) > 0$ such that if $\|\phi\| \le \alpha_1$ we have $\|x_t(t_0,\phi)\| < \beta_1$ for all $t \ge t_0$. This follows from the uniform boundedness already proved. Now choose $\gamma = \gamma(\beta,\alpha) > 0$ so that

$$(8) \qquad e^{-a\gamma(\beta,\alpha)}(b\alpha + c\beta/2(c+d)) \le \frac{\alpha_1}{2},$$

where β in (3) is the uniform bound corresponding to α in the first part of the proof. Since f takes bounded sets into bounded sets there exists an $L > 0$ such that $|f(s,x_t(t_0,\phi)| < L$ for $s \ge t_0$, $\|\phi\| \le \alpha$. Let $k = k(\alpha,\beta)$ be the smallest integer such that

$$k > v(\alpha)/[(\alpha_1/2d)^{w(\alpha_1/4d)}].$$

Suppose now there is a solution $x(t_0,\phi)$ of (2) with $\|\phi\| \le \alpha$ and $\|x_t(t_0,\phi)\| \ge \alpha_1$ for $t \in J \equiv t_0 \le t \le t_0+2(1+k\gamma)$. Since $\alpha_1 > R$, we can consider the function $V(t,x_t(t_0,\phi))$. Condition (i) and (ii) imply that

$$u(|D(t)x_t(t_0,\phi)|) \le V(t,x_t(t_0,\phi)) \le V(t_0,\phi) \le v(\alpha) < u(\beta/2(c+d))$$

so that $|D(t)x_t(t_0,\phi)| < \beta/2(c+d)$ for all $t \in J$. In (7) let $s = s_k$, $t = s_k'$ where $s_k = t_0+(2k-1)\gamma$, $s_k' = t_0+2k\gamma$, $k = 1,2,\ldots,k+1$. Then by Lemma 2,

$$\alpha_1 \le \|x_k'(t_0,\phi)\| \le \{\exp[-a(s_k'-s_k)]\}(b\|\phi\| + c \sup_{t_0 \le u \le s_k'} |D(u)x_u)|$$

$$+ d \sup_{s_k \le u \le s_k'} |D(u)x_u|$$

$$\leq e^{-a\gamma}(ba + c\beta/2(c+d)) + d \sup_{s_k \leq u \leq s_k'} |D(u)x_u|.$$

That is,

$$\alpha_1 \leq \frac{\alpha_1}{2} + d \sup_{s_k \leq u \leq s_k'} |D(u)x_u|.$$

Therefore there must exist a t_k in $[s_u, s_u']$ such that

$$|D(t_u)s_{t_k}| \geq \frac{\alpha_1}{2d}, \quad k=1,2,\ldots,k+1.$$

Since $\|\phi\| \leq \alpha$ and $|f(s,x_s(t_0,\phi))| < L$ for all $s \geq t_0$ on the intervals $I_k = [t_k - \frac{\alpha_1}{4dL}, t_k + \frac{\alpha_1}{4dL}]$ we have $|D(t)x_t| \geq \frac{\alpha_1}{4d}$. As a consequence of this and condition (ii), $\dot{V}(t,x_t) < -w(\alpha_1/4d)$, $t \in I_k$, $k=1,2,\ldots,k+1$. By taking a large L if necessary we can assume that these intervals do not overlap, and hence we have

$$V(t_k, x_{t_k}) \leq V(t_0,\phi) - w(\alpha_1/4d)\frac{\alpha_1}{2d}(k-1) \leq v(\alpha) - w(\alpha_1/4d)\frac{\alpha_1}{2d}(k-1)$$

If $k-1 = K$ then $V(t_k, x_{t_k}(t_0,\phi)) < 0$, which is a contradiction since $u(|D(t_k)x_{t_k}|) > 0$ because $|D(t_k)x_{t_k}| \geq \frac{\alpha_1}{4d} > R$. Therefore there must exist a t' in the interval J such that $\|x_{t'}(t_0,\phi)\| < \alpha_1$. If $t \geq t_0 + T(\alpha)$, $T(\alpha) = 2(1+K(\alpha,\beta)\gamma(\alpha,\beta))$ we have $\|x_t(t_0,\phi)\| < \beta_1$. This completes the proof.

The converse of Theorem 1 can be given under some restrictions of the operator D.

Theorem 3.2. In (1) assume that D satisfies the inequality

$$|D(t)\phi| \leq N\|\phi\|,$$

$\phi \in C$, where $t \geq 0$, and F is locally Lipschitzian in ϕ uniformly in t. Assume that the solutions of (1) are uniform bounded and uniform ultimate bounded. Then there

exists a continuous Liapunov functional $V(t,\phi)$ on $I\times S$ which satisfies the following conditions

(a) $u(|D(t)\phi|) \le V(t,\phi) \le v(\|\phi\|)$,

(b) $\dot{V}(t,\phi) \le -w(|D(t)\phi|)$

where u,v and w have the same properties as in (i) and (ii) of Theorem 1. The condition (b) can be replaced by

(b') $\dot{V}(t,\phi) \le -\alpha V(t,\phi)$, $\alpha > 0$.

(c) V is Lipschitzian, i.e., for any $\phi_1,\phi_2 \in C$, $t \in [0,t_0+T]$, there exists an $M(T) > 0$ such that

$$|V(t,\phi_1) - V(t,\phi_2)| \le M\|\phi_1-\phi_2\|.$$

The proof of Theorem 3.2 needs the following Lemma, which was communicated to the author by Professor Jack Hale.

Lemma 3.1 (Hale). Assume that D in (1) satisfies the inequality

$$|D(t)\phi| \le N\|\phi\|,$$

for $\phi \in C$ where $t \ge t_0$, and F is locally Lipschitzian in ϕ uniformly in t. Then for any $r_0 > 0$ there is a constant $L = L(r_0)$ such that

$$\|x_t(t_0,\phi_1) - x_t(t_0,\phi_2)\| \le e^{L(t-t_0)}\|\phi_1-\phi_2\|$$

for all $t \ge t_0$, and ϕ_1,ϕ_2 for which

$$\|x_t(t_0,\phi_1)\| \le r_0 \qquad \|x_t(t_0,\phi_2)\| \le r_0.$$

Proof of Theorem 3.2. Since the solutions of (1) are uniform ultimate bounded for bound H, say, for any t_0 and any $\alpha > 0$ there exists $T(\alpha)$, such that if $\|\phi\| \le \alpha$ we have $\|x_t(t_0,\phi)\| < H$ for all $t \ge t_0+T(\alpha)$. Since the solutions are uniform bounded, for each $\alpha > 0$ there exists $\beta(\alpha) > 0$ such that if $\|\phi\| \le \alpha$ we have $\|x_t(t_0,\phi)\| \le \beta(\alpha)$, for all $t \ge t_0$. Also there exists $L(\beta(\alpha)) > 0$ such that if $\|\phi_1\| \le \beta$ and $\|\phi_2\| \le \beta$ we have

$$|f(t,\phi_1) - f(t,\phi_2)| \le L(\beta(\alpha))\|\phi_1-\phi_2\|.$$

We can assume $T(\alpha)$, $\beta(\alpha)$, $L(\beta(\alpha))$ are continuous increasing in α. Now define $V(t,\phi)$ as follows

$$V(t,\phi) = \sup_{\tau\ge 0} G(|D(t+\tau)x_{t+\tau}(t_0,\phi)|)e^{\tau}$$

for $0 \le t < \infty$, $\phi \in C$ where

$$G(\xi) = \begin{cases} \xi - H & \xi \ge H \\ 0 & 0 \le \xi < H. \end{cases}$$

This G is a non-negative continuous function for $\xi \ge 0$ and $G(\xi) \to \infty$ as $|\xi| \to \infty$. Also $|G(\xi) - G(\xi')| \le |\xi - \xi'|$. Obviously $G(|D(t)\phi|) \le V(t,\phi)$. We also have

$$V(t,\phi) = \sup_{\tau\ge 0} G(|D(t+\tau)x_{t+\tau}(t_0,\phi)|)e^{\tau} \le \sup_{\tau\ge 0} G(N\|x_{t+\tau}(t_0,\phi)\|)e^{\tau}.$$

From the ultimate boundedness and boundedness assumption, there exist $\mu(\|\phi\|)$ and $\beta(\|\phi\|)$ such that $T(t,\phi) \le \mu(\|\phi\|)$ and $\|x_{t+\tau}(t_0,\phi)\| \le \beta(\|\phi\|)$. Hence $V(t,\phi) \le G(N\beta(\|\phi\|)e^{\mu\|\phi\|} \equiv v(\|\phi\|)$. We have now proved that $G(|D(t)\phi|) \le V(t,\phi) \le v(\|\phi\|)$. One can now take $G(|D(\cdot)\phi|) \equiv u(|D(t)\phi|)$. This verifies a. Since (a)

holds, for any $h > 0$ there exists τ' such that

$$V(t+h,x_{t+h}(t_0,\phi)) = G(|D(t+h+\tau')x_{t+h+\tau'}(t_0,\phi)|)e^{\tau'}$$

Let $\tau = \tau'+h$, then $\tau' = \tau-h$ and

$$V(t+h,x_{t+h}(t_0,\phi)) = G(|D(t+\tau)x_{t+\tau}(t_0,\phi)|e^{\tau}e^{\tau'-\tau},$$

$$\leq V(t,\phi)e^{\tau'-\tau} = V(t,\phi)e^{-h}.$$

Hence

$$V(t+h,x_{t+h}(t,\phi)) - V(t,\phi) \leq -V(t,\phi)[1-e^{-h}].$$

From this we deduce that $\dot{V}(t,\phi) \leq -V(t,\phi) \leq -G(|D(t)\phi|)$, verifying (b) or (b') with $\alpha = 1$.

To show that (c) holds we note that if $t,\tau \in [0,t_0+T]$

$$|V(t,\phi_1) - V(t,\phi_2)|$$

$$\leq \sup_{\tau\geq 0} e^{\tau}|D(t+\tau)x_{t+\tau}(t_0,\phi_1) - D(t+\tau)x_{t+\tau}(t_0,\phi_2)|$$

$$\leq \sup_{T\geq\tau\geq 0} e^{\tau} N\|x_{t+\tau}(t_0,\phi_1) - x_{t+\tau}(t_0,\phi_2)\| ,$$

$$\leq \sup_{T\geq\tau\geq 0} e^{\tau} Ne^{L\beta(\alpha)(t+\tau-t_0)} \|\phi_1-\phi_2\| ,$$

by Lemma 3.1,

$$\leq Ne^{(L\beta(\alpha)+1)(t_0+2T)} \|\phi_1-\phi_2\|.$$

Hence

$$V(t,\phi_1) - V(t,\phi_2) \leq M\|\phi_1-\phi_2\|$$

where $M = Ne^{L(\beta(\alpha)+1)(t_0+2T)}$.

Finally, to prove $V(t,\phi)$ is continuous in t,ϕ we observe that

$$|V(t+h,\phi_0+\phi_1) - V(t,\phi_0)|$$

$$\leq |V(t+h,\phi_0+\phi_1) - V(t+h,x_{t+h}(t,\phi_0+\phi_1))|$$

$$+ |V(t+h,x_{t+h}(t,\phi_0+\phi_1)) - V(t,\phi_0+\phi_1)|$$

$$+ |V(t,\phi_0+\phi_1) - V(t,\phi_0)|.$$

Since V is Lipschitzian,

$$|V(t+h,\phi_0+\phi_1) - V(t+h,x_{t+h}(t,\phi_0+\phi_1))|$$

$$\leq k\|x_t(t,\phi_0+\phi_1) - x_{t_h}(t,\phi_0+\phi_1)\|.$$

Because the solution $x(t;t_0,\phi)$ is continuous given any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|h| \leq \delta$ the right hand side of the above inequality is less than ε. Also

$$|V(t,\phi_0+\phi_1) - V(t,\phi_0)| \leq k\|\phi_1\| < \varepsilon$$

whenever $\|\phi_1\| \leq \delta$. Lastly,

$$|V(t+h,x_{t+h}(t,\phi_0+\phi_1) - V(t,\phi_0+\phi_1)| \leq -V(t,\phi_0+\phi_1)[1-e^{-h}],$$

so that if h is arbitrarily small the right hand side of this inequality can be made very small.

This proves the theorem.

4. On the Boundedness and Periodicity of Certain Scalar Nonlinear Neutral Equations.

The applications in this and next sections are the raison d'etre of the theory presented in Section 3. The calculations involved are less cumbersome than would have been the case if the ideas in [3] were applied. For example, the equation in the transmission line problem, namely,

$$(9) \qquad \frac{d}{dt}[x(t)-qx(t-h)] = -ax(t) - bx(x-h) - \gamma x^3(t) + q\,x^3(t-h) + p(t),$$

where $\gamma > 0$, $|q| < 1$, a,b are constants and p is a ω-periodic function, which was treated in (3), can now be proved to be uniform ultimately bounded by a shorter calculation. Indeed consider $V(t,x) = x^2$ then $u(s) = v(s) = s^2$, so that condition (i) of Theorem 1 holds. It remains only to verify condition (ii).

As in [3], computing the derivative $\dot{V}(Dx_t)$ we have

$$\dot{V}(Dx_t) = -2\gamma D^4\left(\left(1 + \frac{qx(t-h)}{D}\right)^3 - \frac{qx^3(t-h)}{D^3} - \frac{p(t)}{\gamma D^3} + \frac{a}{\gamma D^2} + b + \frac{qax(t-h)}{D^3}\right)$$

If $q = 0$,

$$\dot{V} \leq -\gamma D^4 \text{ if } D \text{ is large.}$$

If $q \neq 0$. By taking R sufficiently large so that $x(t-h)/D < N < \frac{1}{|\sqrt[3]{q}-q|}$ and $|D| \geq R$ we have $\dot{V} < -\gamma\delta D^4$, where

$$\left|-\frac{p(t)}{\gamma D} + \frac{a}{\gamma D^2} + \frac{(b+qa)x(t-h)}{\gamma D^3}\right| \leq \frac{\delta}{2}.$$

We see that condition (ii) holds and since $D\phi = \phi(0) - q\phi(-h)$ is uniformly stable (9) is uniformly ultimate bounded and hence ω-periodic. Observe that the existence of an $f: [R,\infty) \to E'$ and its properties are completely unnecessary here.

The generalized Lienard equation,

$$\ddot{x} + g(t,x)\dot{x} + f(x) = p(t), \tag{10}$$

has had an extensive history. For an excellent summary of results up til 1972, on the continuability, boundedness, oscillation and periodicity of solutions of (10), see Graef [6]. An equation similar to (10) but with history-dependent restoring force is the Lienard equation with delay, namely

$$\ddot{x}(t) + g(x(t))\dot{x}(t) + f(x(t-h)) = 0,\ h > 0. \tag{11}$$

The equation (11) was studied by Hale [7, Section 31] and Grafton [8] who proved the existence of a nontrivial periodic solution, period greater than 2h.

In this section we shall initiate a study of the Lienard equation of neutral type, namely

$$\ddot{x}(t) + a\ddot{x}(t-h) + g(t,x(t)+ax(t-h))\dot{x}(t-h) + f(x+x(t-h)) = p(t,x(t),x(t-h),\dot{x}(t),\dot{x}(t-h)), \tag{12}$$

where $h \geq 1$, $0 < \alpha < 1$, or its equivalent

$$(13) \qquad \frac{d}{dt}(x(t) + \alpha(x(t-h)) = y(t) + \alpha y(t-h)$$

$$\frac{d}{dt}(y(t) + \alpha y(t-h)) = p(t,x(t),x(t-h),y(t)y(t-h)) - f(x+ \alpha x(t-h))$$
$$- g(t,x(t) + \alpha(x(t-h))y(t-h)$$

obtained from (12) on setting

$$\frac{d}{dt} x(s) = y(s).$$

Define the operator D by $D\phi = \phi(0) - A\phi(-h) \equiv \underline{x}(t) - A\underline{x}(t-h)$ where

$$\underline{x}(s) = \begin{matrix} x(s) \\ y(s) \end{matrix},$$

$$A = \begin{bmatrix} -\alpha & 0 \\ 0 & -\alpha \end{bmatrix}$$

Observe that by Lemma 2.1, $D\phi$ is uniformly stable, Indeed the roots of the equation

$$(14) \qquad \det[I - A\rho^{-h}] = 0$$

has modulii less than 1. To see this note that (14) is equivalent to

$$\rho^{h} + \rho = 0.$$

Denote $Dx_t = x(t) + \alpha x(t-h)$, $Dy_t = y(t) + \alpha y(t-h)$ and clearly (13) is the same as

$$(15) \qquad \frac{d}{dt} Dx_t = Dy_t$$

$$\frac{d}{dt} Dy_t = -p(t,x(t),x(t-h)y(t),y(t-h)) - f(Dx_t) - g(t,Dx_t)Dy_t$$

Theorem 4.1. In (15) assume that p, f and g are continuous in their respective arguments, f , g and p take closed and bourded sets into bounded sets. Further assume that

(i) $g(t,\phi) > a > 0$ for $\|\phi\| \geq A > 0$

where a,A are positive constants;

(ii) $f(\psi)\operatorname{sgn}\psi \to \infty$ as $|\psi| \to \infty$;

(iii) there exists a $k > 0$ such that $|p(t,x(t),x(t-h),y(t),y(t-h)| \leq k$ for all $t,x(t),x(t-h)y(t),y(t-h)$. Then (15) is ultimately uniformly bounded. If in addition p is ω-periodic then there exists a ω-period solution of (15).

Proof. Consider the continuous functional

$$V = V_1 + V_2 \tag{16}$$

where

$$2V_1(x_t,y_t) = (Dy_t)^2 + 2\int_0^{Dx_t} f(s)ds \tag{17}$$

$$V_2 = Dx_t \operatorname{sgn} Dy_t \text{ if } |Dx_t| \leq |Dy_t|$$

$$= Dy_t \operatorname{sgn} Dx_t \text{ if } |Dx_t| \geq |Dy_t|.$$

Evidently

$$2V_2 \geq -2|Dy_t|,$$

so that

$$2V \geq 2\int_0^{Dx_t} f(s)ds + (Dy_t)^2 - 2|Dy_t| \tag{18}$$

for all x_t,y_t.

From (18) it is clear that there exists an $H > 0$ such that if $|Dx_t|^2 + |Dy_t| \geq H$ the right hand side of (18) is positive, and condition (i) of Theorem 3.1 holds. Also

$$\dot{V}_{(15)}(x_t,y_t) = -Dy_t g(t,Dx_t)Dy_t + Dy_t p,$$

from which we deduce that

$$\dot{V} \leq -a(Dy_t)^2 + k(Dy_t) \tag{19}$$

if $|Dy_t| \geq A$ and

$$\dot{V} \leq B_1 \text{ if } |Dy_t| \leq A \tag{20}$$

for some constant B_1. Furthermore

(21) $\dot{V}_2 = |Dy_t|$ if $|Dx_t| \leq |Dy_t|$

$= [-g(t,Dx_t)Dy_t - f(Dx_t) + p]\operatorname{sgn}(Dx_t)$

if $|Dx_t| \geq |Dy_t|$ or

(22) $\dot{V}_2 \leq -f(Dx_t)\operatorname{sgn}(Dx_t) + B_2$

if $|Dx_t| \geq |Dy_t|$ for some B_2. Combining (19) and (21), we have that whenever $|Dx_t| \leq |Dy_t|$,

(23) $\dot{V} \leq -a(Dy_t)^2 + (k+1)|Dy_t| \leq -(k+1)|Dy_t|$

provided $|Dy_t| > [\frac{2(k+1)}{a}, A]$, Combining (20) and (22) we have

$$\dot{V} \leq -f(Dx_t)\operatorname{sgn}(Dx_t) + B_1 + B_2$$

whenever $|Dx_t| \geq |Dy_t|$, $|Dy_t| \leq A$,

so that

(24) $\dot{V} \leq -B_3|f(Dx_t)|$

for some $B_3 > 0$ provided $|Dx_t| \geq B_4$. Combining (23) and (24) we deduce that

(25) $\dot{V} \leq \frac{-(k+1)}{2}|Dy_t| - \frac{B_3}{2}|f(Dx_t)|$

provided

$$\min[|Dx_t|, |Dy_t|] \geq \max[A, \frac{2(k+1)}{a}, B_4].$$

Because $|f(\psi)| \to \infty$ as $|\psi| \to \infty$, (25) shows that condition (ii) of Theorem 3.1 is satisfied.

Since D(t) is uniformly stable we deduce from Theorem 3.1 that the solutions of (15) are uniformly bounded and uniformly ultimate bounded.

5. Linear Nonhomogeneous Neutral Equations. We shall study the system of neutral equations described by

$$\frac{d}{dt}[x(t)-Cx(t-h)] - Ax(t) = p(t,x(t),x(t-h)) \tag{26}$$

where $x(s)$ is an n-vector valued function, C and A are $n\times n$ real matrix functions, and p is n-vector valued function which is locally Lipschitzian in the last two arguments and maps bounded sets into bounded sets. We have the following result.

Theorem 5.1. In (26) suppose C is symmetric and all the roots of the equation

(i) $\det[I - C\rho^{-h}] = 0$

have modulii less than 1;

(ii) the eigenvalues λ_i of A satisfy $\lambda_i < -\lambda_0 < 0$, $i=1,2,\ldots,n$;

and those of CAC satisfy $\mu_i > \mu_0 > 0$, $i=1,2,3,\ldots,n$; and the eigenvalues δ_i of $A + CAC$ satisfy $\delta_i < -\delta_0 < 0$, $i=1,2,\ldots,n$.

(iii) The function p satisfies $|p(t,x(t),x(t-h))| < M$, for all $t,x(t)$ $x(t-h)$.

Then the solutions of (26) are uniformly bounded and uniformly ultimately bounded. If, in addition,

(iv) p is ω-periodic then there exists a ω-periodic solution of (26).

Proof. It follows from Lemma 2.1 and condition (i) that the operator $D\phi = \phi(0) - C\phi(-h)$ is uniformly stable. Let

$$V(\phi) = (D\phi,D\phi) + \int_{-h}^{0} (CAC\phi(\theta),\phi(\theta))d\theta, \tag{27}$$

where $(\cdot,\cdot)$ denotes scalar product in E^n. With this definition we can take

$$u(r) = v(r) = V(r),$$

so that condition (i) of Theorem 3.1 is satisfied. Next compute the derivative

of V:

$$\dot{V}(\phi) = (AD\phi, D\phi) + (A\phi(0), \phi(0)) + (CAC\phi(0), \phi(0)) - 2(CAC\phi(-h), \phi(-h)) + 2D\phi p(t, \phi(0), \phi(-h)).$$

Hence

$$\dot{V}(\phi) = (AD\phi, D\phi) + ((A+CAC)\phi(0), \phi(0)) - 2((CAC)\phi(-h), \phi(-h)) + 2D\phi p.$$

By condition (ii),

$$(AD\phi, D\phi) \leq -\lambda_0 |D\phi|^2,$$

$$((A+CAC)\phi(0), \phi(0)) \leq -\delta_0 |\phi(0)|^2,$$

$$-2((CAC)\phi(-h), \phi(-h)) \leq -2\mu_0 |\phi(-h)|^2.$$

By hypothesis (iii)

$$|2D\phi p(t, \phi(0), \phi(-h)| \leq 2M|D\phi|.$$

Hence

$$\dot{V}(\phi) \leq -\lambda_0 |D\phi|^2 - \delta_0 |\phi(0)|^2 - 2\mu_0 |(\phi-h)|^2 + 2M|D\phi|.$$

If we now choose

$$\frac{\lambda_0}{2} |D\phi^2| - 2M|D\phi| > 0,$$

for sufficiently large $|D\phi|$, that is $|D\phi| \geq R$, say, then

$$\dot{V}(\phi) \leq -\frac{\lambda_0}{2} |D\phi|^2. \tag{28}$$

Since (28) holds, condition (ii) of Theorem 1 is satisfied. It now follows that the solutions of (26) are uniformly bounded and uniform ultimate bounded. The existence of a periodic solution follows by Proposition 2.1.

6. Nonlinear System of Neutral Equations. Consider the nonlinear system of ordinary differential equations

$$\dot{x} = f(x,t) \tag{29}$$

where

x,f are n-vectors. Let f_x be the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ and f_x^T its transpose.

The asymptotic behavior of (29) has been investigated under various assumptions on the matrix

$$J(x,t) = \frac{1}{2}[Af_x + f_x^T A]$$

where A is a positive definite symmetric n×n constant matrix. (See Demidovic [9], Ezeilo [10] and [11]). In this section we shall study the system

$$(30) \qquad \frac{d}{dt}[x(t)-Cx(t-h)] = f(t,x(t),x(t-h))$$

in which C is an n×n real constant matrix, x(s) and f are n-vectors; f maps bounded sets into bounded sets and it has continuous partial derivative in the last two arguments. In (30) let D_a, a = 1,2,3 be symmetric n×n matrices

$$\frac{1}{2}(d_{aij} + d_{aji})$$

where

$$d_{1ij} = \frac{\partial f_i}{\partial \phi_j(0)}(t,\phi(0),\phi(-h))$$

$$d_{2ij} = \frac{\partial f_i}{\partial \phi_j(-h)}(t,0,\phi(-h))$$

$$d_{3ij} = \frac{\partial f_i}{\partial \phi_j(0)}(t,\phi(0),0)$$

Let

$$J_a = AD_a + D_a^T A, \quad i=1,2,3$$

where A is a positive definite symmetric n×n constant matrix. We shall prove the boundedness of solutions of (30) under certain restrictions on J_a. Let $D\phi = \phi(0) - C\phi(-h)$. Then (30) is equivalent to the system

$$(31) \qquad \frac{d}{dt}[D\phi] = f(t,\phi(0),\phi(-h)).$$

We assume that D is a uniformly stable operator.

Theorem 6.1. In (30), suppose C is symmetric and all the roots of the equation

$$\det[I - Cp^{-h}] = 0$$

have modulii less tnan 1. Let A be a real positive definite symmetric n×n-constant matrix such that the eigenvalues λ_{1i} of the matrix J_1 satisfy

(i) $\lambda_{1i} \leq -\delta_1 < 0$, $|\phi(0)| \geq R$, $|\phi(-h)| \geq R$,

$\lambda_{1i} \leq \Delta_1$, $(0 < \Delta_1 < \infty)$, $|\phi(0)| \leq R$, $|\phi(-h)| \leq R$,

for all $t \geq 0$; and

(ii) the eigenvalues λ_{4i} of the matrix $-J_1C$, where C is in (30) satisfy

$\lambda_{4i} < -\delta_4 < 0$, $|\phi(0)| \geq R$, $|\phi(-h)| \geq R$,

$\lambda_{4i} < \Delta_4$, $(0 < \Delta_4 < \infty)$ $|\phi(0)| \geq R$, $|\phi(-h)| \geq R$.

Furthermore,

(iii) $\|J_2\| \leq \Delta_2$, for all t and $\phi(-h)$, $\|-J_3C\| \leq \Delta_3$, for all t and $\phi(0)$,

where the constants $\delta_1, \delta_4, \Delta_2, \Delta_3$ are such that

(32) $$a_3 \equiv \frac{3\delta_4}{8} - \frac{\delta_1}{8}\|C\|^2 > 0,$$

$$a_3 \frac{\delta_1}{8} \geq a_2^2,$$

where $a_2 \equiv \frac{\delta_1}{4}\|C\| - (\Delta_2+\Delta_3)$.

(iv) Suppose also that $|f(t,0,0)| \leq M$, for all t. Then the solutions of (30) are unfiromly bounded and uniform ultimate bounded. If in addition f is ω-periodic there exists a nontrivial ω-periodic solution of (30).

Remark. In (iii), $\|\cdot\|$ denotes matrix norm.

The proof of the Theorem rests on the following Lemma, a version of which proved very useful in Ezeilo [10] and Chukwu [12].

Lemma 3. Let $g: E^1 \times E^n \times E^n \to E^n$ be a function such that $g(t,\phi(0),\phi(-h)$ and $\frac{\partial g_i}{\partial \phi_j(0)}$,

$1 \leq i,j \leq n$ are continuous in t, $\phi(0)$, and $\phi(-h)$. Suppose there is a constant M such that $-\infty < M$, and the characteristic roots of the matrix

$$\frac{1}{2}\left(\frac{\partial g_i}{\partial \phi_j(0)} + \frac{\partial g_j}{\partial \phi_i(0)}\right)$$

satisfy

$$v_k \leq M, \quad k=1,\ldots,n$$

uniformly in the arguments. Then the scalar product g defined by

$$G = (g(t;\phi(0)+\phi_i h,\phi(-h)) - g(t;\phi(0),\phi(-h),h)$$

satisfies

$$g \leq M|h|^2 \text{ for all } t.$$

Proof of Theorem. Consider the function V defined by

(32) $$V(\phi) = (AD\phi,D\phi),$$

where $D\phi = \phi(0) - C\phi(-h)$ and $(\cdot,\cdot)$ denotes the inner product in E^n. Since A is positive definite V is obviously a positive definite quadratic form in $D\phi$. In fact there are positive finite constants n_1,n_2 such that

(33) $$n_1|D\phi|^2 \geq V(\phi) \geq n_2|D\phi|^2.$$

Take $u = n_2|D\phi|^2$, $v = n_1|D\phi|^2$ in condition (i) of Theorem 3.1. Now compute the derivative $\dot{V}_{(30)}(\phi)$ and note that A is symmetric:

$$\begin{aligned}
\dot{V}(\phi) &= (Af(t,\phi(0),\phi(-h)),D\phi) + (AD\phi,f(t,\phi(0),\phi(-h)) \\
&= 2(Af(t,\phi(0),\phi(-h)),D\phi) \\
&= 2(Af(t,\phi(0),\phi(-h)),\phi(0)-C\phi(-h)) \\
&= 2(Af(t,\phi(0),\phi(-h)),\phi(0)) + 2(Af(t,\phi(0),\phi(-h),-C\phi(-h)) \\
&\equiv N_1 + N_2.
\end{aligned}$$

Now

$$N_1 = 2U_1 + 2U_2 + 2U_3,$$

where

$$U_1 = (Af(t,\phi(0),\phi(-h)) - Af(t,0,\phi(-h)),\phi(0))$$

$$U_2 = (Af(t,0,\phi(-h)) - Af(\iota,0,0),\phi(0))$$

$$U_3 = (Af(t,0,0),\phi(0)).$$

Also

$$N_2 = 2U_4 + 2U_5 + 2U_6$$

where

$$U_4 = (Af(t,\phi(0),\phi(-h))- Af(t,\phi(0),0),-C\phi(-h))$$

$$U_5 = (Af(t,\phi(0),0) - Af(t,0,0),-C\phi(-h))$$

$$U_6 = (Af(t,0,0),-C\phi(-h))$$

By applying Lemma 3 to U_i, i=1,2,4,5, we deduce that

$$U_1 = (J_1\phi(0),\phi(0)) \leq -\frac{\delta_1}{2}|\phi(0)|^2 + \beta_1$$

$$U_4 = (-J_1C\phi(-h),\phi(-h)) \leq -\frac{\delta_4}{2}|\phi(-h)|^2 + \beta_4,$$

for all $\phi(0),\phi(-h)$ and t, where β_1,β_4 depend only on $\delta_1,\delta_4,\Delta_1,\Delta_4$ and R appearing in (i) and (ii). Also by (iii)

$$U_2 = (J_2\phi(-h),\phi(0)) \leq \Delta_2(|\phi(0)||\phi(-h)|)$$

$$U_5 = (-J_3C\phi(-h),\phi(0)) \leq \Delta_3(|\phi(0)||\phi(-h)|).$$

Because A and C are constant matrices and f(t,0,0) is bounded,

$$U_3 \leq \delta_3|\phi(0)|,$$

$$U_6 \leq \delta_6|\phi(-h)|,$$

where δ_3 and δ_6 depend only on A, C, and M in (iv).

On gathering all these estimates we obtain

$$\dot{V}(\phi) \leq -\frac{\delta_1}{2}|\phi(0)|^2 - \frac{\delta_4}{2}|\phi(-h)|^2 + (\Delta_2+\Delta_3)(|\phi(0)|\cdot|\phi(-h)| + \delta_3|\phi(0)| + \delta_6|\phi(-h)| + \beta_1 + \beta_4. \tag{34}$$

The first three terms on the right hand side of this inequality can be recast in the form

$$-a_0\|D\phi\|^2 - a_1|\phi(0)|^2 - 2a_2|\phi(0)||\phi(-h)| - a_3|\phi(-h)|^2$$
$$- \frac{\delta_1}{4}|\phi(0)|^2 - \frac{\delta_4}{8}|\phi(-h)|$$

where

$$\|D\phi\|^2 = [|\phi(0)| - \|C\|\,|\phi(-h)|]^2$$

and

$$a_0 = \frac{\delta_1}{8}, \qquad a_1 = \frac{\delta_1}{8}$$
$$a_2 = \frac{\delta_1}{4}\|C\| - (\Delta_2+\Delta_3)$$
$$a_3 = \frac{3\delta_4}{8} - \frac{\delta_1}{8}\|C\|^2$$

Observe that $a_0 > 0$, $a_1 > 0$. Also by (iii), $a_3 > 0$ and $a_1 a_3 \geq a_2^2$. Hence

$$\dot{V}(\phi) \leq -a_0[|D\phi|]^2 - \frac{\delta_1}{8}|\phi(0)|^2 - \frac{\delta_4}{8}|\phi(-h)| + \delta_3|\phi(0)| + \delta_6|\phi(-h)|$$
$$+ \beta_1 + \beta_2.$$

Hence for $|\phi(0)|$ and $|\phi(-h)|$ large, say, $|\phi(0)| \geq R$, $|\phi(-h)| \geq R$,

$$\dot{V}(\phi) \leq -a_0[|D\phi|]^2.$$

Condition (ii) of Theorem 1 is met. Since D is uniformly stable the theorem follows.

III. Boundedness of Ordinary and Hereditary Systems of Lurie Type

1. Introduction

In the problem of Lurie

(i) $$\dot{x} = Ax + bf(\sigma),$$
$$\dot{\sigma} = c^t x - rf(\sigma),$$

where A is an n x n constant matrix whose characteristic roots have negative real parts, x, b and c are n-vectors and σ, f and r are scalars, one tries to find a necessary and sufficient conditions for the absolute stability of (i). One major approach deals with showing that a certain Liapunov function, whose existence is guaranteed by the uniform asymptotic stability of the equation

(ii) $$\dot{x} = Ax,$$

is positive definite with negative definite derivatives with respect to (i). This then yields conditions for bounded solutions to be uniformly asymptotically stable. In this procedure, independent arguments are needed to show that solutions are bounded. [13]. In a more recent paper Burton [14] took the significant step of attacking the problem of boundedness directly and independently of the Lurie problem. When $f(\sigma)/\sigma \to 0$ as $|\sigma| \to \infty$ he obtains necessary and sufficient conditions for all solutions of (i) to be uniformly ultimately bounded.

In this chapter we consider plant equations of more general types, say nonlinear ordinary differential equations, nonlinear functional equations of retarded and neutral types. We assume that the uncontrolled system is uniform-bounded and uniform ultimate bounded and then use the inverse theorems ensuring the existence of Liapunov functionals for the controlled system, ([15], [1], and Chapter II) and obtain sufficient conditions for the uniform boundedness and uniform ultimate boundedness of the feedback system.

2. Notations, Definitions and Preliminary Results.

The following notations will be used in this paper: E^n denotes the Euclidean n-space and the norm of any $x \in E^n$ is written as $|x|$. For any $h \geq 0$, C denotes the space of continuous functions mapping $[-h,0]$ into E^n with the sup norm denoted by $\|\cdot\|$. For any continuous function $x(\theta), -h \leq \theta \leq A$, $A>0$ and fixed t, $0 \leq t \leq A$, x_t denotes the function $x_t(\theta) = x(t+\theta)$, $-h \leq \theta \leq 0$.

Let $t_0 \in E$, and g be a continuous function taking $I \times C$, $I = [t_0 \infty]$ into E^n. Assume that g is linear in ϕ and that there exists an n x n matrix function $\mu(t,\theta)$ $t \in I$ $\theta \in [-h,0]$ which is of bounded variation in θ and there exists a scalar function $\ell(\theta)$, continuous and non-decreasing for $s \in [0,h]$, $\ell(0)=0$ such that

$$(1) \qquad g(t,\phi) = \int_{-h}^{0} [d_\theta \mu(t,\theta)]\phi(\theta)$$

$$\left| \int_{-s}^{0} [d_\theta \mu(t,\theta)]\phi(\theta) \right| \leq \ell(s) \sup_{-s \leq \theta \leq 0} |\phi(\theta)|$$

for all $t \in I$, $\phi \in C$.

Define a functional differential operator

$$D(\cdot): \quad I \times C \to E^n,$$

by

$$(2) \qquad D(t)\phi = \phi(0) - g(t,\phi), \quad t \in I, \ \phi \in C.$$

We shall study the following differential equations:

$$(3) \qquad \frac{dx}{dt} = A(t,x) \qquad ;$$

$$(4) \qquad \frac{dx}{dt} = A(t,x) + bf(\sigma) \quad ,$$

$$\frac{d\sigma}{dt} = B(t,x) - rf(\sigma) \quad ;$$

$$(5) \qquad \frac{dx}{dt} = F(t,x_t) \qquad ;$$

$$(6) \qquad \frac{dx}{dt} = C(t,x_t) + bf(\sigma) \quad ,$$

$$\frac{d\sigma}{dt} = E(t,x_t) - rf(\sigma) \quad ;$$

and

$$\frac{d}{dt}\,(D(t)x_t) = F(t,x_t) \quad ; \tag{7}$$

$$\frac{d}{dt}\,(D(t)x_t) = F(t,x_t) + bf(\sigma), \tag{8}$$

$$\frac{d\sigma}{dt} = G(t,D(t)x_t) - rf(\sigma).$$

Here A is a continuous function from $I\times C\to E^n$, F is a continuous function from $I\times C\to E^n$. Both A and F are Lipschitzian in their second arguments respectively. Also $x, b \in E^n$; f, r and σ are scalars. B is a continuous scalar function from $I\times E^n \to E^1$, E: $I\times C\to E^1$, and G: $I\times E^n\to E^1$. D is as defined in (2).

We now define the boundedness concepts we need for the different systems.

Definition 2.1. (i) The solutions $x(t,x_o)$ of (3) are said to be uniform-bounded, if for any $\alpha>0$, there exists a $\beta(\alpha)$ such that if $|x_o|\leq\alpha$ we have $|x(t,x_0)| \leq \beta(\alpha)$ for all $t \geq t_0$.

(ii) The solutions $x(t,x_0)$ of (3) are said to be uniform-ultimate-bounded for bound B, if there exists a positive constant B and for any $\alpha>0$ there exists a $T(\alpha)$ such that if $|x| \leq\alpha$ we have $|x(t,x_0)| \leq B$ for all $t \geq t_0 + T(\alpha)$. Here $x(t,x_0)$ denotes the solution of (3) with $x(t_0,x_0) = x_0$. In the same way one can define the same concepts for the functional differential equation (7). We shall denote the solution of (5) or (7) by $x(t_0,\phi)$ if $x_{t_0}(t_0,\phi) = \phi$, the initial function ϕ is assumed to belong in C.

For example, the solutions of (7) are uniform-bounded if for any $\alpha>0$ there exists a $\beta(\alpha)$ such that if $\|\phi\| \leq \alpha$, we have $\|x_t(t_0,\phi)\| \leq \beta(\alpha)$ for all $t\geq t_0$.

In Cruz and Hale [5] the concept of a uniformly stable operator was

introduced and was shown to imply the following:

Definition 2.2. The operator D in (2) is uniformly stable if there are constants β , $\alpha>0$ such that the solution $x(t_0,\phi)$ of the "difference equation"

$$D(t)x_t = 0, \quad t \geq t_0$$

$$x_{t_0} = \phi, \quad D(t_0)\phi=0,$$

satisfies $\|x_t(t_0,\phi)\| \leq \beta \ell^{-\alpha(t-t_0)} \|\phi\|$,

for all $t \geq t_0$.

If $V: I \times C \to E$ is continuous we define the "derivative" $\dot{V}(t,\phi)$ along solutions of (7), say, as

$$\dot{V}(t,\phi) = \dot{V}_{(7)}(t,\phi) = \lim_{h \to 0} \frac{1}{h} [V(t,x_{t+h}(t_0,\phi)) - V(t,\phi)]$$

Definition. We say $F(t,x) \in C_0(t,x)$ if for any compact set $\hat{E} \subset E^1 \times E^n$, there exists a constant $K(\hat{E})$ such that, for any pair of points $(t,x) \in \hat{E}$, $(t^1,x^1) \in \hat{E}$, we have

$$|F(t,x) - F(t^1,x^1)| \leq K[|t-t^1| + |x-x^1|]$$

where K is independent of t and may depend on x.

Moreover, we shall say that $F(t,x) \in C_0(x)$ if for any compact set $\hat{E} \subset E^n$, there exists a constant $K(\hat{E},t)$ such that for any pair of points $x \in \hat{E}$, $x^1 \in \hat{E}$,

$$|F(t,x) - F(t,x^1)| \leq K|x-x^1|.$$

If K is independent of t we say

$$F(t,x) \in \bar{C}_0(x)$$

The following converse theorems are reproduced from [15], [1], and Chapter II.

Theorem 2.1. Consider the system (3) in which $A \in C_0(x)$. If the solutions of (3) are uniform bounded and uniform ultimate-bounded for bound B^1, there

exists a Liapunov function $V(t,x)$ defined on I, $\|x\| \geq R$, $R>0$ such that

(i) $a(|x|) \leq V(t,x) \leq b(|x|)$, for $|x| \geq R$, $t \in I$,

where $a(r)$ is continuous increasing $a(r)>0$ for $r \geq R$ and $a(r) \to \infty$ as $r \to \infty$; $b(r)$ is continuous increasing, $b(r) \to \infty$ as $r \to \infty$;

(ii) $\dot{V}(t,x) \leq -cV(t,x)$, $c>0$,

with $R>B$[1].

(iii) If A is bounded for $|x|$ bounded, and if $A \in \bar{C}_0(x)$ then $V(t,x) \in \bar{C}_0(t,x)$ with Lipschitz constant M.

Theorem 2.2. [1] In (5) assume that for any $\alpha>0$ there exists an $L(t,\alpha)>0$ such that if $\|\phi\| \leq \alpha$, we have

(9) $$|F(t,\phi)| \leq L(t,\alpha)$$

where $L(t,\alpha)$ is continuous in t; and that $F(t,\phi) \in \bar{C}_0(\phi)$. Let S be the set of $\phi \in C$ such that $\|\phi\| \geq H$, where H is a positive constant which may be large.

If the solutions of (5) are uniform-bounded and uniform ultimate bounded there exists a continuous Liapunov functional $V(t,\phi)$ on $I \times S$ which satisfies

(i) $a(\|\phi\|) \leq V(t,\phi) \leq b(\|\phi\|)$,

where $a(r)$ is continuous, increasing, positive for $r>H$ and $a(r) \to \infty$ as $r \to \infty$, $b(r)$ is continuous and increasing,

(ii) $\dot{V}_{(5)}(t,\phi) \leq \alpha V(t,\phi)$, $\alpha>0$,

α a constant

(iii) $V(t,\phi) \in \bar{C}_0(\phi)$.

Theorem 2.3. (Chapter II). Consider the system (7) where $F(t,\phi)$ is locally Lipschitzian in ϕ, and where the uniformly stable operator D satisfies

$$|D(t)\phi| \leq N\|\phi\|, \text{ for all } t \in I,$$

$\phi \in C$ with K the Lipschitz constant for F. If the solutions of (7) are uniform-bounded and uniform ultimate-bounded, there exists a continuous Liapunov functional $V(t,\phi)$ on $I \times S$ which satisfies the following conditions

(i) $u(|D(t)\phi|) \leq V(t,\phi) \leq v(\|\phi\|)$

for all $t \in I$, $\phi \in C$, where $u(r) > 0$ for $r \geq H$ and $a(r) \to \infty$ as $r \to \infty$; $v(r)$ is continuous increasing and $v(r) \to \infty$ as $r \to \infty$;

(ii) $V(t,\phi) \leq -\omega(|D(t)\phi|)$

or

(ii) $V(t,\phi) \leq -\alpha V(t,\phi)$, $\alpha > 0$

(iii) V is locally Lipschitzian, i.e. for any ϕ_1, $\phi_2 \in C$, $t \in [0, t_0+T]$ there exists some M(T) such that

$$|V(t,\phi_1) - V(t,\phi_2)| \leq M\|\phi_1 - \phi_2\|, \text{ if } \|\phi_i\| \leq r, \text{ some } r.$$

Remark. In the above theorems if the relevant functions are assumed to be Lipschitzian i.e., for example, for each x, $x^1 \in E^n$

$$|F(t,x) - F(t,x^1)| \leq K|x-x^1|$$

then the Liapunov functionals produced are also Lipschitzian but the Lipschitz constant M may be dependent on t. Throughout what follows we shall denote the Lipschitz constant by M.

Define the set S as follows

$$S = \{\phi \in C: \|\phi\| \geq H, \text{ H may be large.}\}$$

We now reproduce the following theorem from Chapter II which we shall need.

Theorem 2.4. Suppose there exists a continuous functional V: $[t_0^{\infty}] \times S$ such that:

(i) $u(|D(t)\phi|) \leq V(t,\phi) \leq v(\|\phi\|)$,

where u(r) is continuous increasing positive for $r > H$ and $u(r) \to \infty$ as $r \to \infty$, and v(r) is continuous and increasing;

(ii) $\dot{V}_{(7)}(t,\phi) \leq -\omega(|D(t)\phi|)$,

where $\omega(r)$ is continuous and positive for $r>H$.

If D(t) is uniformly stable then the solutions of (7) are uniform-bounded and uniform ultimate bounded.

Sufficient conditions for ultimate boundedness were earlier given by Lopes [3] in terms of the so called Liapunov-Razumikhin functions. The conditions are far from necessary and the treatment is not complete.

3. Boundedness Theorems.

Theorem 3.1. Consider the uncontrolled ordinary differential equation (3), and the controlled system (4), where A, B, c, b, f, and r are identified in Section 2, and where it is assumed that A and C and f Lipschitzian in σ, A is bounded for $|x|$ bounded. Suppose that the solutions of (3) are uniform-bounded and uniform ultimate bounded . Let a, c, M be given by Theorem 2.1. Assume that

(i) $\int_0^{\sigma} f(s)ds \to \infty$ as $|\sigma| \to \infty$,

$f(0)=0$, $\sigma f(\sigma)>0$ if $\sigma \neq 0$;

(ii) The scalar function $C(t,x)$ is continuous in x and t and is such that

$$|C(t,x)| \leq a(|x|),$$

where a is given by Theorem 2.1

(iii) Suppose the constants $C>0$, $M>0$ given be Theorem 2.1 satisfy

$$4cr > (M|b| + 1)^2$$

Then the controlled system (4) is uniform bounded and uniform ultimate bounded.

Proof. Let $V(t,x)$ be the Liapunov function guaranteed for $\|x\| \geq R$ by

Theorem 2.1, and let $\dot{V}$ be its derivative along solutions of (4). Then

$$\dot{V} \leq -cV + M|bf(\sigma)| \ .$$

Define

$$W = \frac{V^2}{2} + \int_0^{\sigma} f(s)\, ds.$$

Because of the properties of V, obviously there exists functions $\bar{a}(r)$ and $\bar{b}(r)$ with properties as indicated in condition (i) of Theorem 2.1 such that

$$\bar{a}(|x|) + \int_0^{\sigma} f(s)ds \leq W(x,\sigma) \leq \bar{b}(|x|) + \int_0^{\sigma} f(s)ds$$

Because of condition (i) of Theorem 3.1, condition A and B of Yoshizawa's Theorem 7 [3 p 86] are satisfied for the function W.

The derivative of W along solutions of (4) satisfies

$$\dot{W} \leq -cV^2 - r|f(\sigma)|^2 + VM|b||f(\sigma)| + |f(\sigma)||c(t,x)|$$

On using hypothesis (ii), we obtain

$$\dot{W} \leq -cV^2 + V(M|b| + 1)|f(\sigma)| - r|f(\sigma)|^2$$

The condition (iv) makes this quadratic form in V and $|f(\sigma)|$ negative definite for $|x| \geq R$. Hence, property (C) of Theorem 7 in [15] is also satisfied. It follows from Theorem 3 and Theorem 7 of [15] that the solutions of (4) are uniform-bounded and uniform ultimate bounded.

Remark. The function W was inspired by a recent paper of Somolinos [16] on the absolute stability of the system (5).

Theorem 3.2. Consider the nonlinear functional differential equation (6) where F, E, f, b, r are identified in Section 2. Assume that the functions F, E and f are Lipschitzian in x_t, and σ respectively. Suppose that three functions satisfy conditions similar to (9) in Theorem 2.2. Assume that the uncontrolled system (5) is uniform-bounded and uniform ultimate bounded. Let a, α and M be as determined by Theorem 2.2. Assume that

(i) $f(0) = 0$, $\sigma f(\sigma) > 0$, $\sigma \neq 0$;

$$\int_0^{\sigma} f(s)ds \to \infty \text{ as } |\sigma| \to \infty;$$

(ii) $|E(t,\phi) \leq a(\|\phi\|)$, where $a(r)$ is continuous, increasing positive for $r>H$ and $a(r) \to \infty$ as $r \to \infty$;

(iii) $4\alpha r > (M|b| + 1)^2$.

Then the solutions of the feedback system is uniformly bounded and uniform ultimate bounded.

Theorem 3.2 is proved in the same way as Theorem 3.1 is proved. One uses Theorem 2.2 to obtain a Liapunov functional V. The function

$$W = \frac{V^2}{2} + \int_0^{\sigma} f(s)ds$$

has the required properties of V in Theorem 1 in [1]. This Theorem of Yoshizawa now ensures that the solutions of (6) are uniform-bounded and uniform-ultimate-bounded.

Theorem 3.3. Consider the system (8), where the uniformly stable operator D satisfies

$$|D(t)\phi| \leq N\|\phi\|$$

for all $t \in I$, $\phi \in C$, where K is the Lipschitz constant for F. The functions $F(t,\phi)$, $f(\sigma)$, $g(t,D(t)\phi)$ are Lipschitzian in ϕ, σ and $D(t)\phi$, respectively. Suppose the system (7) is uniform-bounded and uniform-ultimate-bounded. Let u, α and M be as given by Theorem 2.3. Assume that

(i) $f(0)=0$ $\sigma f(\sigma)>0$ $\sigma \neq 0$, $\int_0^{\sigma} f(s)ds \to \infty$ as $|\sigma| \to \infty$;

(ii) $|g(t,D(t)\phi)| \leq u|D(t)\phi|$ for all $t \in I$, $\phi \in C$

where $u(r)$ is continuous, increasing positive for $r>H$ and $u(r) \to \infty$ as $r \to \infty$; the relation

(iii) $$4\alpha r > \left[\frac{M|b|}{1-\ell(\theta)} + 1\right]^2$$

holds for all $\theta \in [0,h]$.

Then the solutions of the feed back system (8) are uniform-bounded and Uniform-ultimate bounded.

Proof. Since solutions of (7) are uniform-bounded and uniform-ultimate bounded, Theorem 2.3 guarantees the existence of a Liapunov functional V on $I^{\times}S$. Differentiating V along solutions of (8) we deduce that

$$\dot{V}_{(8)}(t,\phi) \leq \dot{V}_{(7)}(t,\phi) + M \overline{\lim_{h\to 0}} \frac{1}{h} [y_{t+h}(t,\phi) - x_{t+h}(t,\phi)] \tag{10}$$

where $x = x(t,\phi)$, $y = y(t,\phi)$ are solutions of (7) and (8) respectively. Also,

$$D(t+h)(y_{t+h}-x_{t+h}) = \int_t^{t+h} bf(\sigma)ds,$$

for any $h \geq 0$. Since $g(t,\phi)$ satisfies (1) there is a $h_0>0$ such that

$$|y_{t+h} - x_{t+h}| \leq \frac{1}{1-\ell(h_0)} \int_t^{t+h} bf(\sigma)ds$$

for $0 \leq h \leq h_0$. On using (10), one obtains

$$\dot{V}_8(t,\phi) \leq \dot{V}_{(7)}(t,\phi) + \frac{M}{1-\ell(h_0)} |bf(\sigma)|, \tag{11}$$

for all $t \geq t_0$ $\phi \in C$. Hence,

$$\dot{V}_8(t,\phi) \leq -\alpha V(t,\phi) + \frac{M}{1-\ell(h_0)} |bf(\sigma)|. \tag{12}$$

Now define

$$W = \frac{V^2}{2} + \int_0^{\sigma} f(s)ds.$$

Differentiate W with respect to t along solutions of (8) to obtain

$$\dot{W} \leq -\alpha V^2 - r|f(\sigma)|^2 + \frac{MV}{1-\ell(h_0)} |bf(\sigma)|$$

$$+ |f(\sigma)||G(\cdot,v(t)x_t)|$$

$$\leq -\alpha V^2 - r|f(\sigma)|^2 + V\left[\frac{M|b|}{1-\ell(h_0)} + 1\right] |f(\sigma)|,$$

where we have used hypothesis (ii). Thus

$$\dot{W} \leq -\alpha V^2 + V\left[\frac{M|b|}{1-\ell(h_0)} + 1\right] |f(\sigma)| - r|f(\sigma)|^2.$$

This quadratic function in V and $|f(\sigma)|$ is negative definite because of condition (iii). It is simple to verify that W satisfies conditions (i)-(ii) of Theorem 2.4. Because D is uniformly stable, the operator $\overline{D}$ defined by

$$\overline{Dx}_t = \begin{bmatrix} D(t)x_t \\ \sigma \end{bmatrix},$$

where D(t) is given in (2) and x_t and σ are as above, is uniformly stable. It follows now that the equation (8) is uniform-bounded and uniform ultimate bounded.

IV. Absolute Stability of Neutral Functional Differential Equation of Lurie Type

1. Introduction.

Consider a system of real ordinary differential equations

$$\frac{dx}{dt} = Ax + bf(\sigma) \tag{1}$$

$$\frac{d\sigma}{dt} = c^T x - rf(\sigma)$$

in which $f: (-\infty,\infty) \to (-\infty,\infty)$ is sectionally continuous with $\sigma f(\sigma)>0$ for $\sigma\neq 0$, $f(0)=0$, A is an n x n matrix, c and b are constant n-vectors and r is a scalar. The problem of Lurie consists of finding a necessary and sufficient condition for every solution $(\sigma(t),x(t))$ of (1) to tend to (0,0) as $t\to\infty$ whenever it is assumed that the uncontrolled equation

$$\frac{dx}{dt} = Ax \tag{2}$$

is uniformly asymptotically stable in the large (cf [17, p 9]). The entire monograph by Lefschetz was devoted to this problem. Recently, Somolinos [16] has generalized this problem of Lurie to functional differential equation of retarded type. In this chapter we shall treat the problem of Lurie when the system is described by functional differential equation of neutral type. We shall assume that the uncontrolled system is uniformly asymptotically stable. Utilizing a Theorem of Cruz and Hale in [5] which ensures the existence of a Liapunov functional, we then obtain conditions for the uniform asymptotic stability of the feedback system.

2. Notations and Preliminary Results.

Let E^n be a real n-dimensional Euclidean vector space with norm $|\cdot|$. Let $h \geq 0$ be a given real number. Let C be the space $C([-h,0],E^n)$ of continous functions taking $[-h,0]$ into E^n with $\|\phi\|$, $\phi \in C$ defined by $\|\phi\| = \sup\{|\phi(\theta)| : -h \leq \theta \leq 0\}$. For any continuous function $x(\theta)$ on $-h \leq \theta \leq t_1$, $t_1 > 0$ and a fixed t, $0 \leq t \leq t_1$, x_t denotes the function $x_t(\theta) = x(t+\theta)$, $-h \leq \theta \leq 0$. Let $D(\cdot)\colon [t_0,\infty) \times C \to E^n$ be a continuous function defined by

$$D(t)\phi = \phi(0) - g(t,\phi), \text{ for } t \in [t_0,\infty) \equiv I, \phi \in C, \tag{3}$$

where $g\colon [t_0,\infty) \times C \to E^n$,

is continuous, $g(t,\phi)$ is linear in ϕ and is given by

$$g(t,\phi) = \int_h^0 [d_s\mu(t,s)]\phi(s). \tag{4}$$

The function $\mu(t,s)$ is an n x n matrix $t \in I$, $s \in [-h,0]$, with elements of bounded variation in s which satisfy the following condition:

$$\left| \int_\theta^0 [d_s\mu(t,s)]\phi(s) \right| \leq \ell(\theta) \sup_{-\theta \leq r \leq 0} |\phi(r)|, \tag{5}$$

for all $t \in I$, $\phi \in C$, where ℓ is continuous nondecreasing for $\theta \in [0,h]$, $\ell(0)=0$

Let $A\colon I \times C \to E^n$ be continuous and consider the equation

$$\frac{d}{dt}(D(t)x_t) = A(t,x_t), \tag{6}$$

$$x_{t_0} = \phi, \; t_0 \in I.$$

The following theorem ensures the existence of a Liapunov functional when (6) is uniformly asymptotically stable.

Theorem 2.1 [5]. Let D(t) and $A(t,\cdot)$ be bounded linear operators from C into E^n such that for some constant $L>0$, for all $\phi \in C$, for all $t \geq t_0$,

$$|D(t)\phi| \leq L\|\phi\|.$$

If (6) is uniformly asymptotically stable, then there exist positive constants M, α and a continuous scalar function V on $I \times C$ such that

(7) (i) $|D(t)\phi| \leq V(t,\phi) \leq M\|\phi\|$,

(ii) $\dot{V}(t,\phi) \leq -\alpha V(t,\phi)$,

(iii) $|V(t,\phi)-V(t,\psi) \leq K\|\phi-\psi\|$,

for all $t \geq t_0$, $\phi, \psi \in C$; $\dot{V}$ is the usual upper right hand derivate along the solutions of (6).

In Theorem 2.1 it is assumed that D(t) and $A(t,\cdot)$ are linear. However, Cruz and Hale [5] stated a similar result when $A(t,\phi)$ is not linear in ϕ, but $g(t,\phi)$ in (3) satisfies

$$|g(t,\phi) \leq L\|\phi\|, \text{ for all } t \geq t_0.$$

We now state the result and point out the required lemma needed to carry out the proof in [5]. It was communicated to the author by Professor J. K. Hale.

Theorem 2.2. Let $A(t,0) = 0$, and let $A(t,\phi)$ be uniformly locally Lipschitzian in ϕ uniformly with respect to t, with Lipschitz constant N. Let D satisfy locally the condition:

$$|D(t)\phi| \leq K|\|\phi\|,$$

for all $t \geq t_0$, for some K.

Assume that the null solution of (6) is uniformly asymptotically stable. Then there exists a $S_0 > 0$, a $M = M(S_0) > 0$, positive definite functions b(u), c(u), on $0 \leq u \leq S_0$ and a scalar function $V(t,\phi)$ defined and continuous for $t \in I \times C$, $\|\phi\| \leq S_0$ such that

(a) $|D(t)\phi| \leq V(t,\phi) \leq b(\|\phi\|)$

(b) $\dot{V}(t,\phi) \leq -c(|D(t)\phi|)$

(c) $|V(t,\phi_1) - V(t,\phi_2)| \leq M\|\phi_1 - \phi_2\|$

for all $t \geq t_0$, ϕ_1, $\phi_2 \in C$, $\|\phi_i\| \leq S_0$, i=1,2. The condition (b) can be replaced by

(b') $\dot{V}(t,\phi) \leq -\beta V(t,\phi)$, $\beta>0$.

Remark. The problem with the proof of Theorem 7.2 in [5] is contained in verifying (c). The following lemma is needed.

Lemma. (Hale) In (6) assume that D satisfies the conditions of Theorem 2.2. Then for any r_0 0, there is a constant $L=L(r_0)$ such that

$$\|x_t(t_0,\phi_1) - x_t(t_0,\phi_2)\| \leq e^{L(t-t_0)} \|\phi_1 - \phi_2\|$$

for all $t \geq t_0$, ϕ_1, ϕ_2 for which

$$\|x_t(t_0,\phi_1)\| \leq r_0 \qquad \|x_t(t_0,\phi_2)\| \leq r_0.$$

Remark. The proof is not as easy as for retarded equations since one cannot apply the Gronwall inequality directly one must take small steps in time and make careful estimates using the properties of D(t).

To prove Theorem 2.2, set

$$\dot{V}(t,\phi) = \sup_{s\geq 0} |D(t+s)x_{t+s}(t_0,\phi)| e^{q(t)},$$

and proceed as in page 310 of Hale [18]. Our lemma replaces the inequality on page 310, bottom line.

The first case considered is the indirect control system

(8) $$\frac{d}{dt}(D(t)x_t) = A(t,x_t) + bf(\sigma), \quad t \geq t_0,$$

$$\dot{\sigma} = B(t,D(t)x_t) - rf(\sigma),$$

$$x_{t_0} = \phi, \quad t_0 \in I,$$

in which A is as above, C(t,y) is a scaler continuous function in $t \geq 0$, $y \in E^n$, and f is a scalar function which is continuous.

Definition. The operator D in (3) is uniformly stable if there are constants $\alpha > 0$, $\beta > 0$ such that the solution of the "difference equation"

$$D(t)x_t = 0 \quad ,$$

$$x_{t_0} = \phi \quad ,$$

$$D(t_0)\phi = 0 \quad ,$$

satisfies $\|x_t\| \leq \beta e^{-\alpha(t-t_0)}\|\phi\|$, $t \geq t_0$.

3. Main Theorems.

Theorem 3.1. Assume that in (8) the uncontrolled system (6) is uniformly asymptotically stable. Let α, and K be as given by Theorem 2.1. Assume that $A(t,\cdot)$ and $D(t)$ are bounded linear operators from C into E^n such that $|D(t)\phi| \leq M\|\phi\|$ for all $t \geq t_0$, $\phi \in C$. Assume that:

(i) $\int_0^{\sigma} f(s)ds \to \infty$, as $|\sigma| \to \infty$;

there exists a positive constant c such that

(ii) $|B(t,D(t)\phi)| \leq c(|D(t)\phi|)$,

for all $t \in I$, $\phi \in C$;

(iii) for all $\theta \in [0,h]$ the relation

$$4\alpha r > \left(c + \frac{K(b)}{1-\ell(\theta)}\right)^2 \quad ,$$

holds where ℓ is defined in (5);

(iv) the operator D is uniformly stable.

Then (8) is uniformly asymptotically stable.

Proof. Since (6) is uniformly asymptotically stable, there exists a Liapunov functional for (6) given by Theorem 2.1. Let $\dot{V}_{(8)}$ denote the derivative of V along the solutions of (8). Let $y = y(t_0,\phi)$, $x = x(t_0,\phi)$ be the solutions of (8) and (6) respectively, then the relations (7) imply that

$$\dot{V}_{(8)}(t,\phi) \leq \dot{V}_6(t,\phi) + K \lim_{h\to 0} \frac{1}{h} |y_{t+h}(t,\phi) - x_{t+h}(t,\phi)| \tag{9}$$

But then

$$D(t+h)(y_{t+h}-x_{t+h}) = \int_t^{t+h} bf(\sigma)ds,$$

for any $h \geq 0$. Since g satisfies (5) we have that there exists an $h_0>0$ such that

$$|y_{t+h}-x_{t+h}| \leq \frac{1}{1-\ell(h_0)} \int_t^{t+h} |bf(\sigma)|ds,$$

for $0 \leq h \leq h_0$. We now use this inequality in (9) to obtain

$$\dot{V}_8(t,\phi) \leq \dot{V}_{(6)}(t,\phi) + \frac{K}{1-\ell(h_0)} |bf(\sigma)|. \tag{10}$$

Hence, by (7(ii))

$$\dot{V}_8(t,\phi) \leq -\alpha V + \frac{K}{1-\ell(h_0)} |bf(\sigma)|.$$

Define $W = \frac{V^2}{2} + \int_0^{\sigma} f(s)ds$.

The derivative of W along the solutions of (8) satisfies

$$\dot{W} \leq -\alpha V^2 - r|f(\sigma)|^2 + V\left(\frac{K}{1-\ell(h_0)}|bf(\sigma)|\right) + |f(\sigma)c|.$$

By conditions (ii) of Theorem 3.1 and (i) of Theorem 2.1 we obtain from this that

$$\dot{W} \leq -\alpha V^2 - r|f(\sigma)|^2 + V\left(\frac{K|b|}{1-\ell(h_0)} + c\right)|f(\sigma)| \tag{11}$$

The right hand side of (11) is a quadratic form in V and $|f(\sigma)|$. It is obviously negative definite by condition (iii). Hence, there exists a

positive number γ such that

$$\dot{W} \leq -\gamma(V^2 + |f(\alpha)|^2)$$

From this it follows that

$$\dot{W} \leq -\gamma|D(t,\phi)|^2,$$

so that the second condition of (4.2) in Theorem 4.1 of Cruz-Hale [5] is met for the Liapunov function W. Trivially, also all the other conditions in (4.2) are satisfied. Because D is a uniformly stable operator the operator $\bar{D}$ given by

$$\bar{D}\psi = \psi(0) - \bar{g}(t,\psi),$$

$$\text{where } \psi = \begin{bmatrix} \psi \\ \sigma \end{bmatrix},$$

$$\bar{g} = \begin{bmatrix} g \\ 0 \end{bmatrix},$$

is uniformly stable. Therefore, by Theorem 4.1 of [5] the system

$$\frac{d}{dt}(\bar{D}(t)g_t) = \bar{g}(t,g_t)$$

is uniformly asymptotically stable.

Here

$$\bar{D}(t)y_t = \begin{pmatrix} D(t)x_t \\ \sigma \end{pmatrix}$$

$$\bar{g}(t,y_t) = \begin{bmatrix} A(t,x_t) + bf(\sigma) \\ B(t,D(t)x_t) - rf(\sigma) \end{bmatrix}$$

The proof is complete.

Theorem 3.2. Consider (8), and assume that $A(t,0) = 0$, $A(t,\phi)$ is locally Lipschitzian in ϕ uniformly with respect to t, and the operator D satisfies

$$|D(t)\phi| \leq M\|\phi\|,$$

locally in $\phi \in C$, for all $t \geq t_0$ and some M. Assume that D is uniformly stable and that (6) is uniformly asymptotically stable. Let K and β be as given by Theorem 2.2 and assume that

(i) For all $\theta \in [0,h]$ the relation

$$4\beta r > \left(c + \frac{K|b|}{1-\ell(\theta)}\right)^2$$

holds where ℓ is defined in (5), and where c is a constant such that

(ii) $|B(t,D(t)\phi)| \leq c|D(t)\phi|$,

for all $t \in I$, $\phi \in C$:

(iii) $\int_0^{\sigma} f(s)ds \to \infty$, as $|\sigma| \to \infty$.

Then there exists a $\delta_0>0$, such that for any ε, $0 < \varepsilon < \delta_0$ and any $t_0 \geq 0$, there is a $\delta=\delta(\varepsilon)$ such that $\|\phi\| < \delta$ implies $\|x_t(t_0,\phi)\| < \varepsilon$ for all $t \in [t_0,\infty]$, and for any $\eta>0$, $0 \leq \eta \leq \delta_0$, there exists a $T(\eta)>0$, such that $\|\phi\| \leq \delta$, implies $\|x_t(t_0,\phi)\| \leq \eta$, if $t \geq t_0 + T(\eta)$. In other words all solutions in the ball $S(\delta_0) \subseteq C$ are uniformly asymptotically stable.

Proof. The hypotheses of the theorem imply there is a Liapunov functional V satisfying the conditions of Theorem 2.2. Choose δ_0 as in Theorem 2.2. Let $\dot{V}_{(8)}$ denote the derivative of V along solutions of (8). If $y = y(t_0,\phi)$, $x = x(t_0,\phi)$ are the solutions of (8) and (6) respectively, then, as before,

$$\dot{V}_8(t,\phi) \leq \dot{V}_6(t,\phi) + \frac{K}{1-\ell(h_0)} |bf(\sigma)|,$$

provided $\|\phi\| \leq \delta_0$. Our using

$$W = \frac{V^2}{2} + \int_0^{\sigma} f(s)ds,$$

one easily verifies that the conditons of Theorem 4.1 of [5] are satisfied for W, provided $\|\phi\| \leq \delta_0$. By the cited Theorem the trivial solutions of (8) is uniformly asymptotically stable when confined to the ball $S(\delta_0) \subseteq C$.

Consider the direct control case:

$$\frac{d}{dt}(D(t)x_t) = A(t,x_t) + bf(\sigma), \tag{12}$$

$$\sigma = c^T D(t)x_t,$$

$$x_{t_0} = \phi$$

where the letters are there defined above and $c^T b = -r<0$.

Theorem 3.3. Assume that $D(t)$ and $A(t,\cdot)$ are bounded linear operators from C into E^n, such that

$$|D(t)\phi| \leq L\|\phi\|$$

for all $t \geq t_0$, $\phi \in C$, and

$$|A(t,\phi)| \leq a|D(t)\phi|, \quad a>0. \tag{13}$$

Suppose (6) is uniformly asymptotically stable and

(i) $f(0) = 0$, $\sigma f(\sigma)>0$ $\sigma \neq 0$, f continuous and

$$\int_0^\sigma f(s)ds \to \infty, \text{ as } |\sigma| \to \infty.$$

(ii) Let α and K be given by Theorem 2.1 and let the relation

$$4\alpha r > \left(\frac{K|b|}{1-\ell(s)} + a|c|\right)^2, \tag{14}$$

hold for all $s \in [0,h]$, where ℓ is defined in (5).

Then (12) is uniformly asymptotically stable.

Proof. Proceed as before, using Theorem 2.1 to obtian a Liapunov functional V for the system 6. Differentiating V along solutions of (12) yields

$$\dot{V}_{(12)}(t,\phi) \leq V_{(6)}(t,\phi) + \frac{K}{1-\ell(h_0)}|b||f(\sigma)|.$$

Set

$$W = \frac{V^2}{2} + \int_0^\sigma f(s)ds.$$

Then

$$\dot{W}_{(12)} \leq -\alpha V^2 - r|f(\sigma)|^2 + \frac{VK|bf(\sigma)|}{1-\ell(h_0)}$$

$$+ |f(\sigma)||c^T A(t,x_t)|$$

$$\leq -\alpha V^2 - r|f(\sigma)|^2$$

$$+ V\left[\frac{k|b|}{1-\ell(h_0)} + a|c|\right] |f(\sigma)|$$

where we have used (13) and the property of V. We now use (14) to deduce the result as before.

V. Functional Inclusion and Controllability of Nonlinear Functional Differential Systems

1. Introduction.

In this chapter we formulate sufficient conditions for the existence of a solution of a nonlinear differential inclusion of neutral type,

$$\frac{d}{dt} D(t,x_t) \in R(t,x_t), \tag{1}$$

where D is a continuous operator on I×C, linear in x_t, indeed of the form (4) below, and $R(\cdot,\cdot)$ denotes a set valued mapping of I×C into the set of non-empty closed convex subsets of E^n. The solution is required to satisfy an initial and terminal condition

$$x_{t_0} = \phi_0, \; x_{t_1} = \phi_1, \tag{2}$$

where $\phi_0,\phi_1 \in C$, C a function space. The theory includes functional differential inclusions with delay, treated in [19] and ordinary differential inclusions treated in [20] It is related to the existence result in [21]. Our proof uses the Fan fixed point theorem in [22].

As a consequence of the existence result, we present sufficient conditions for the exact function space controllability of the nonlinear neutral control system

$$\frac{d}{dt} D(t,x_t) = f(t,x_t,u), \; u(t) \in \Omega(t,x_t). \tag{3}$$

We give explicit conditions on D, f and Ω which guarantee exact controllability between two fixed functions. The equation (3) includes those studied by Cruz and Hale in [5]. They are general enough to include systems of the form

$$x(t) - \sum_{i=1}^{k} A_i(t)x(t-h_i) = \sum_{i=1}^{k} B(t)x(t-h_i) + g(t,x(t-h_i),\cdots,x(t-h_k),u)$$

Our work in controllability generalizes the treatment in [19] and [20]. Our viewpoint is different from the recent investigations in [23] by Jacobs and Langenhop of the system

$$\dot{x}(t) = A_{-1}x(t-h) + A_0x(t) + A_1x(t-h) + Bu(t).$$

Their studies are more algebraic and their controls are unrestricted.

2. Definitions and Notations.

Let E^n be a real n-dimensional Euclidean vector space with norm $|\cdot|$. Let $h \geq 0$ be a given real number. We shall denote by C the space $C([-h,0],E^n)$ of continuous functions taking $[-h,0]$ into E^n with $\|\phi\|$, $\phi \in C$, defined by $\|\phi\| = \sup_{-h\leq\theta\leq 0} |\phi(\theta)|$. Also D will denote the space $C([t_0,t_1],E^n)$, and I will represent the interval $[t_0,t_1]$. If $x: [t_0-h,t_1] \to E^n$, $h > 0$, then for $t \in I$ the symbol x_t denotes the function on $[-h,0]$ defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-h,0]$.

Let G denote an open subset of $I\times C$ and let $D: G \to E^n$ be a given continuous function. We shall be interested in the differential inclusion

$$\frac{d}{dt} D(t,x_t) \in R(t,x_t),$$

with $x_{t_0} = \phi_0$, $x_{t_1} = \phi_1$, $\phi_0,\phi_1 \in C$, where $D(t,\phi)$ is linear in ϕ and is given by

$$D(t,\phi) = \int_{-h}^{0} [d_s A(t,s)]\phi(s). \tag{4}$$

The function $A(t,s)$ is an $n\times n$ matrix with elements of bounded variation in $s \in [-h,0]$ which satisfy the following condition

$$B(t) = A(t,0) - A(t,0^-), \quad \det B(t) \neq 0, \tag{5}$$

$$\left|\int_{-h}^{0} [d_s A(t,s)]\phi(s) - B(t)\phi(0)\right| \leq \gamma(t,h)\|\phi\|,$$

$(t,\phi) \in G$, where $B(t)$ is continuous and $\gamma(t,h)$ is continuous for $t \in [t_0,\infty)$, $h \geq 0$, $\gamma(t,0) = 0$. The mapping R above is set-valued and maps $I\times C$ into the set of nonempty closed and convex subsets of E^n. A function x is said to be a solution of (1) if $x \in C([t_0-h,t_1],E^n)$ $(t,x_t) \in G$ and x satisfies (1) on I.

In this definition it is $D(t,x_t)$ and not $x(t)$ which is continuously differentiable on I. For $\phi \in C$ we say $x(t,\phi)$ is a solution of (1) through (t_0,ϕ_0) if $x(t,\phi)$ is a solution of (1) on $[t_0-h,t_1]$ and $x_{t_0}(t_0,\phi_0) = \phi_0$.

We shall present sufficient conditions which guarantee the existence of a solution of (1) which satisfies the boundary values (2). We shall then apply this to study the controllability of the system

$$\frac{d}{dt} D(t,x_t) = f(t,x_t,u),\ u(t) \in \Omega(t,x_t), \tag{6}$$

where f is continuous in all its arguments, and D is given in (4) and satisfies (5). A function x is said to be a solution of (6) through (t_0,ϕ_0) if $x(t_0,\phi_0,u)$ is a solution of (6) on $[t_0-h,t_1]$ and $x_{t_0}(t_0,\phi_0,u) = \phi_0$. Note that x is a solution of (6) through (t_0,ϕ_0) if and only if x satisfies the equation

$$D(t,x_t) = D(t_0,\phi_0) + \int_{t_0}^{t} f(s,x_s,u(s))ds,\ t \in I, \tag{7}$$

$$x_{t_0} = \phi_0. \tag{8}$$

For the results in the existence of solutions of neutral functional differential equations defined by (6) and (4), see [24]-[25].

We now introduce the following notations. Let $m(t) \geq 0$, $t_1 \leq t \leq t_2$ be a given scalar function, $m \in L_1([t_1,t_2],E^1)$ and let $\mathfrak{C}$ be the set of all continuous functions $x: [t_0-h,t_1] \to E^n$ such that $x_{t_0} = \phi_0$, $x_{t_1} = \phi_1$, where $\phi_0,\phi_1 \in C$, and such that $D(t,x_t)$, $t \in I$ is continuously differentiable and have derivatives satisfying

$$|\frac{d}{dt} D(t,x_t)| \leq m(t) \text{ a.e. on } I. \tag{9}$$

Let C_p be a compact ball in C of radius p. For sufficiently large p, C_p can be chosen nonempty.

3. Existence of Solutions.

In this section, we give sufficient condition on R to ensure the existence of a solution of the initial value problem (1) and (2).

Theorem 3.1. Consider the generalized boundary value problem

$$\frac{d}{dt} D(t,x_t) \in R(t,x_t) \text{ on } I, \tag{10}$$

$$x_{t_0} = \phi_0, \; x_{t_1} = \phi_1,$$

$\phi_1, \phi_2 \in C$ where D is defined in (4) and satisfies (5); and has ker $D(t,\cdot) = \{0\}$; and R denotes a set-valued mapping of $I \times C$ into the set of nonempty compact convex subsets of E^n. Suppose R possesses the following properties:

(i) R is upper semicontinuous with respect to inclusion; that is, for every $\beta > 0$ there exist δ_i, i=1,2, such that the inequalities

$$|t-\bar{t}| < \delta_1, \; \|x_t - x_{\bar{t}}\| < \delta_2,$$

imply $R(t,x_t) \subseteq U_\beta$, where U_β is the closed β-neighborhood of $R(\bar{t},x_{\bar{t}})$.

(ii) The relation

$$D(t_1,\phi_1) - D(t_0,\phi_0) \in \int_{t_0}^{t_1} R(s,x_s)ds$$

holds for all $x \in C_p$.

(iii) For each measurable y(t) satisfying the inclusion $y(t) \in R(t,x_t)$, where $x(t) \in C_p$, we have

$$|y(t)| \leq m(t) \text{ a.e. on } I, \text{ with } m \in L_1(I,E^1).$$

Then the generalized boundary value problem has at least one solution $x \in C_p$.

The following set-valued map Φ on C_p is needed in the proof of Theorem 3.1 and is defined by

$$\Phi(y) = \{z \in C_p : \frac{d}{dt} D(t,z_t) \in R(t,y_t), \; t \in I\}. \tag{11}$$

Note that $z \in C_p$ implies that $z_t \in C$. The next two lemmas are crucial in our proof.

Lemma 3.1. Assume conditions (ii) and (iii). If $y \in C_p$, then $\phi(y)$ is non-empty and convex-valued.

Proof of Lemma 3.1. The convexity of $\phi(y)$ follows from those of C_p and $R(t,y_t)$ and the linearity of $D(t,y_t)$ on y_t. Now let $y \in C_p$, then from condition (ii), there exists a measurable function σ such that $\sigma(t) \in R(t,y_t)$ a.e. on I, such that

$$D(t_1,\phi_1) = D(t_0,\phi_0) + \int_{t_0}^{t_1} \sigma(s)ds.$$

For some $z_t \in C$, with $\max\{|z(s)| : t_0-h \leq s \leq t_1\} \leq p$, set

$$D(t,z_t) = D(t_0,\phi_0) + \int_{t_0}^{t} \sigma(s)ds,$$

and note that

$$\frac{d}{dt} D(t,z_t) = \sigma(t) \in R(t,y_t) \text{ a.e. on I.}$$

Because $y \in C_p$, and $\sigma(t) \in R(t,y_t)$

$$\left|\frac{d}{dt} D(t,z_t)\right| = |\sigma(t)| \leq m(t),$$

by condition (iii). Also

$$D(t_1,z_{t_1}) = D(t_0,\phi_0) + \int_{t_0}^{t_1} \sigma(s)ds = D(t_1,\phi_1),$$

so that $z_{t_1} = \phi_1$, since ker $D(t,\cdot) = \{0\}$. Similarly

$$D(t_0,z_{t_0}) = D(t_0,\phi_0),$$

so that

$$z_{t_0} = \phi_0.$$

Hence $z \in C_p$, so that $z \in \Phi(y)$, and $\Phi(y)$ is nonempty. This proves the Lemma.

Lemma 3.2. Assume that the conditions of Theorem 3.1 are satisfied. Then Φ defined in (11) has a closed graph; that is, suppose $\{y^n\} \subseteq C_p$ and assume $z^n \in \Phi(y^n)$ for $n=1,2,\cdots$. If $z^n \to z \in C_p$ and $y^n \to y \in C_p$ then

$$z \in \Phi(y).$$

Proof of Lemma 3.2. Because $z^n \in \Phi(y^n)$

$$\frac{d}{dt} D(t,z_t^n) \in R(t,y_t^n), \quad t \in I.$$

Since $y_t^n \in C_p$, by condition (iii),

$$|\dot{D}(t,z_t^n)| \leq m(t),$$

where $m \in L_1(I,E^1)$. From the above inequality we deduce that

$$|D(t,z_t^n) - D(t_0,\phi_0)| \leq \int_{t_0}^{t} m(s)ds,$$

and

$$|D(t,z_t^n) - D(\bar{t},z_{\bar{t}}^n)| \leq \int_{\bar{t}}^{t} m(s)ds.$$

So that

$$\int_{E_i} \dot{D}(s,z_s^n)ds \to 0 \text{ when } i \to \infty,$$

uniformly with respect to n for each decreasing sequence $\{E_i\}$, $E_i \subseteq I$, with void intersection. Therefore (see [27], p. 292), there is a sequence (we retain the same notation) weakly convergent in $L_1(I,E^n)$ to a function $\zeta \in L_1(I,E^n)$. Then for each $t \in I$,

$$D(t,z_t) = \lim_{n\to\infty} D(t,z_t^n) = \lim_{n\to\infty} [D(t_0,\phi_0) + \int_{t_0}^{t} \dot{D}(s,z_s^n)ds$$

$$= D(t_0,\phi_0) + \int_{t_0}^{t} \zeta(s)ds.$$

It now follows from page 422 of [27] that there is a sequence $\{\zeta_k\}$ of convex combinations of the function $\{\dot{D}(t,z^k),\dot{D}(t,z_t^{k+1}),\cdots\}$ converging in $L_1(I,E^n)$ norm to ζ. From this sequence $\{\zeta_k\}$ select a subsequence which converges to ζ a.e. Thus almost everywhere on I,

$$\zeta(t) \in \bigcap_{k=1}^{\infty} \overline{co}\Big(\bigcup_{n=k}^{\infty} \dot{D}(t,z_t^n)\Big) \subseteq \bigcap_{k=1}^{\infty} \overline{co}\Big(\bigcup_{n=k}^{\infty} R(t,y_t^n)\Big) \subseteq R(t,y_t), \tag{12}$$

where $\overline{co}(M)$ is the closed convex hull of $M \subseteq E^n$. Hence

$$\frac{d}{dt} D(t,z_t) \in R(t,y_t),$$

and Φ has a closed graph. The argument given here is completely analogous to that in [28] and [29]. The Lemma may also be proved by using the reasoning in [19] in this case we need only assume that R is closed valued and not necessarily compact. Note that we used the upper semicontinuity of R in (12).

We now reproduce the Fan fixed point theorem of [22] on which our proof of Theorem 3.1 rests.

Proposition 3.1. Let L be a locally convex topological linear space and k a compact convex set in L. Let W(k) be the family of all closed convex (nonempty) subsets of k. Then for any upper semicontinuous point-to-set transformation f from k into W(k), there exists a point $x_0 \in k$ such that $x_0 \in f(x_0)$. Here upper semicontinuity means that limit $x_n = x_0$, $y_n \in f(x_n)$ and limit $y_n = y_0$ implies $y_0 \in f(x_0)$.

Proof of Theorem 3.1. Observe that the set of all continuous functions on $[t_0-h,t_1]$ into E^n equipped with the sup norm is a locally convex topological linear space and that C_p is a chosen compact convex subset of this. Also Φ defined in (11) is a mapping of C_p into the set of subsets of C_p which has a closed graph and is upper semicontinuous with respect to set inclusion. Since

C_p is compact and ϕ has closed values, $\phi(y)$ is compact for $y \in C_p$. Since C_p and $R(t,y_t)$ are convex and $D(t,y_t)$ is linear in y_t, ϕ is convex-valued. Apply Proposition 3.1 to deduce a fixed point $x^0 \in \phi(x^0)$; that is $x^0 \in C_p$, $\frac{d}{dt} D(t,x^0_t) \in R(t,x^0_t)$, $x^0_{t_0} = \phi_0$, $x^0_{t_1} = \phi_1$. The function x_0 is the desired solution.

4. Controllability.

In this section, we apply the existence theorems for Section 3 to study the controllability of nonlinear neutral functional differential systems:

$$\frac{d}{dt} D(t,x_t) = f(t,x_t,u) \text{ on } I, \tag{13}$$

where $D: I\times C \to E^n$ is continuous and linear in x_t and, as assumed in (5), atomic at 0. We shall assume as basic that the ker $D(t,\cdot)$ is $\{0\}$. Also the function f in (13) is a mapping $f: I\times C\times E^m \to E^n$ which is continuous in all its arguments. The control set is a multi-function $\Omega: I\times C \to E^m$ with values $\Omega(t,\phi)$ nonempty, compact subsets of E^m, which is upper semicontinuous with respect to set inclusion. Let $\mathcal{U}(\Omega)$ be the set of all measurable selections $u: I \to E^m$ with $u(t) \in \Omega(t,x_t)$ for each $t \in I$. It is well-known that $\mathcal{U}(\Omega) \neq \emptyset$ (see [30], p. 398). The system (13) is controllable if given $\phi_0,\phi_1 \in C$, there exists a $u \in \mathcal{U}(\Omega)$ such that the solution $x(t,\phi_0,u)$ of (13) passing through (t_0,ϕ_0) satisfies $x_{t_1}(\cdot,\phi_0,u) = \phi_1$. The next result states a sufficient condition for (13) to be controlled from one function to another.

Theorem 4.1. Let $\phi_0,\phi_1 \in C$. (i) Assume that the set

$$R(t,\phi) = \{f(t,\phi,u): u \in \Omega(t,\phi),\ t \in I,\ \phi \in C\} \tag{14}$$

is convex. Furthermore, assume that (ii) the relation

$$D(t_1,\phi_1) - D(t_0,\phi_0) \in \int_{t_0}^{t_1} R(s,x_s)ds,$$

holds for all $x \in C_p$. Also

(iii) $|f(t,x_t,u)| \leq m(t)$, $m \in L_1(I,E^1)$, for each $t \in I$, $x \in C_p$ and $u \in \Omega(t,\phi)$.

Then there exists a $u \in \mathcal{U}_{ad}(\Omega)$ such that the solution $x(t,t_0,\phi_0,u)$ of (13) satisfies

$$x_{t_0}(\cdot,t_0,\phi_0,u) = \phi_0,\ x_{t_1}(\cdot,t_0,\phi_0,u) = \phi_1$$

Proof of Theorem 4.1. Because f is continuous and Ω is upper semicontinuous with respect to set inclusion and has compact values, R is upper semicontinuous and has compact values. By (i) R is convex. Condition (ii) is the same as (ii) in Theorem 3.1 for R defined in (14). Condition (iii) here implies hypothesis (iii) of Theorem 3.1. Since all the conditions of Theorem 3.1 are satisfied the existence of a solution x of the generalized boundary value problem

$$\frac{d}{dt} D(t,x_t) \in R(t,x_t) \text{ on } I, \tag{15}$$

$$x_{t_0} = \phi_0,\ x_{t_1} = \phi_1, \tag{16}$$

follows from the theorem. It remains to verify that every solution of (15) that in addition satisfies (16) can be viewed as a trajectory of (13) with (16) holding. That this is the case follows from the well-known ideas of Filippov [31] which were extended to cover our situation by McShane and Warfield [32]. This result in [32] was later applied in a way similar to us by Angell in [33]. The existence of a $u \in \mathcal{U}_{ad}(\Omega)$ which generates $x(t;\ t_0,\phi_0,u)$ such that (16) holds is now established.

Corollary 4.1. Consider the system

$$\dot{x}(t)-A_1\dot{x}(t-h) = B_1x(t) + B_2x(t-h) + Cu, \tag{17}$$

where the coefficient matrices are constants and the operator $D\phi = \phi(0)-A_1\phi(-h)$ has {0} as kernel. Let the set $\Omega(t) \subseteq \mathcal{U}$ be closed and convex and upper semi-

continuous with respect to inclusion, where $\mathcal{U}$ is compact. Let $\phi_0, \phi_1 \in C$, and assume that

(i) $$\phi_1(0) - \phi_0(0) + A_1\phi_0(-h) - A_1\phi_1(-h) \in R,$$

where

$$R = \int_{t_0}^{t_1} \{B_1 x(s) + B_2 x(s-h) + C\Omega(s)\}ds.$$

Then there exists a $u \in \mathcal{U}_h(\Omega)$ such that the solution $x(t)$ of (17) satisfies (16).

Remark. The controllability of the system (17) was recently investigated by Jacobs and Langenhop in [23]. It was assumed there in [23] that the control set is unrestrained.

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VI. Controllability of Delay Systems with Restrained Controls

I. Introduction.

Consider the control system

$$\dot{x}(t) = L(t,x_t) + K(t,u), \quad t \geq t_0,$$
$$x(t) = \phi \quad \forall\, t \in [t_0-h, t_0], \tag{1}$$

where $L(t,\phi)$ is continuous in t, linear in ϕ and is given explicitly by

$$L(t,\phi) = \sum_{k=1}^{\infty} A_k(t)\phi(-t_k) + \int_{-\tau}^{0} A(t,\xi)\phi(\xi)d\xi\,, \tag{2}$$

where each $A_k(t)$, $A(t,\xi)$ are continuous n x n matrix functions for $-\infty<t,\xi<\infty$ $0 \leq t_k$, $\tau \leq h$. It is assumed as basic that $K(t,u)$ is continuous in t and u.

Numerous contributions have been made on the controllability of (1) when power available in (1) are unlimited and the controls are allowed to be any square integerable functions on $[t_0^{\infty}]$ with values in E^m. In [34]-[37] function space controllability was investigated, while in [38], Euclidean space controllability was studied.

The purpose of this paper is to consider both the Euclidean and the function space controllability of (1) when the available control power is limited and the controls have values restricted to compact and convex subsets of a Euclidean space, E^m. The unifying theme of the present chapter is the introduction of a growth condition which for systems without delay was extensively used in [39]-[41]. A system is asymptotically proper if, and only if it possesses this growth condition. In the appropriate space a system is asymptotically proper if and only if it is controllable. As a consequence, we show that under rather mild conditions, the system (1) is controllable if, and only if its perturbations are controllable.

The controls are square integrable with values in P, a compact convex subset of E^m. The state of the system is either E^n or the function space $W_2^{(1)}$. It is sometimes convenient to use the larger space $C=([-h,0],E^n)$ of continuous function from $[-h,0] \to E^n$.

2. Preliminaries.

Let E^n be a real n-dimensional Euclidean vector space with norm $|\cdot|$. Let $h \geq 0$ be a given real number. We shall use $W_2^{(1)}$ to denote the Sobolev space $W_2^{(1)}([-h,0],E^n)$ of functions whose derivatives are square integrable. This space is a Hilbert space with inner product defined by

$$(x,y) \equiv (x(-h),y(-h))_{E^n} + \int_{-h}^{0} (\dot{x}(t),\dot{y}(t))_{E^n}\, dt$$

for $x,y \in W_2^{(1)}([-h,0],E^n)$, where $(\cdot,\cdot)_{E^n}$ denotes the inner product in E^n.

Let $h>0$ be given. If $x: [t_0-h,t_1]\ E^n$ then for $t \in [t_0,t_1]$ the symbol x_t denotes the function on $[-h,0]$ defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-h,0]$. Let P be a closed bounded convex subset of E^m and let

$$\mathbb{P} = \{u: u \in L_2([t_0,t_1],P)$$

Set $I \equiv [t_0 t_1]$. Throughout what follows the constraint set P could be replaced be a sphere or cubic in E^m, or the family $\{P(t): t \in I$ of closed convex sets in E^m which are contained in a sphere in E^m.

In the last case,

$$\mathbb{P} = \{u: u \in L_2([t_0,t_1],E^m)\ u(t) \in P(t)\}.$$

We shall assume in the sequel that the L in equation (1) satisfies the following condition:

$$|L(t,\phi)| \leq \ell(t)\|\phi\|, \quad t \in [t_0^{\infty}), \quad \phi \in C$$

where ℓ is such that

$$\int_t^{t+h} \ell(s)ds \leq \ell_1, \quad t \in [t_0,\infty),$$

ℓ_1 a constant. We shall also assume that $K(t,u)$ is continuous in t and $u \in IP$.

Under the above assumptions the solution of (1) is given by

$$x(t;t_0,\phi,u) = x(t;t_0,\phi,0) + \int_{t_0}^{t} U(t,s)K(s,u(s))ds, \quad t \geq t_0 \tag{3}$$

$$x_{t_0} = \phi \text{ in } [t_0-h,t_0].$$

Where U satisfies the equation

$$\frac{\partial}{\partial t} U(t,s) = L(t,U_t(\cdot,s) \quad t \geq s \tag{4}$$

$$U(t,s) = \begin{cases} 0 & s-h \leq t < s \\ I & t = s \end{cases}$$

if $U_t(\cdot,s)(\theta) = U(t+\theta,s)$ $-h \leq \theta \leq 0$ we can write the solution as follows

$$x_t(t_0,\phi,u)(\theta) = x_t(t_0,\phi,0)(\theta) + \int_{t_0}^{t} U_t(\cdot,s)(\theta)K(s,u(s))ds \tag{5}$$

$t \geq t_0$ $\theta \in [-h,0]$; or,

$$x_t(t_0,\phi,u) = x_t(t_0,\phi,0) + \int_{t_0}^{t} U_t(\cdot,s)K(s,u(s))ds$$

Throughout the paper, the initial function $\phi(\theta) \equiv 0$ $\theta \in [t_0-h,t_0]$, so that, since $x_t(t_0,\phi,0)$ is linear in ϕ, (see page 82 of Ref 9)

$$x_t(t_0,\phi,u) = \int_{t_0}^{t} U_t(\cdot,s)K(s,u(s))ds. \tag{6}$$

In Euclidean space the solution is

$$x(t,t_0,0,u) = \int_{t_0}^{t} U(t,s)K(s,u(s))ds \tag{7}$$

Definition. The Euclidean reachable set of (1) at time t is the subset of E^n given by

$$IR(t,t_0) = \{\int_{t_0}^{t} U(t,s)K(s,u(s))ds: u \in IP\}$$

The reachable set is

$$\mathbb{R}(t_0) = \bigcup_{t \geq t_0} \mathbb{R}(t,t_0).$$

Definition. The system (1) is Euclidean controllable if, and only if $\mathbb{R}(t_0)=E^n$. Equivalently, (1) is Euclidean controllable if and only if for each $x_1 \in E^n$, there exists a time $t_1 \geq t_0$ and a control $u \in \mathbb{P}$ such that the solution $x(t,t_0,0,u)$ of (1) satisfies $x(t_1;t_0,0,u) = x_1$.

Definition. The function space reachable set of (1) at time t is the subset of $W_2^{(1)}$ given by

$$\mathfrak{C}(t,t_0) = \{\int_{t_0}^{t} U_t(\cdot,s)k(s,u(s))ds: u \in \mathbb{P}\} .$$

The function space reachable set is

$$\mathfrak{C}(t_0) = \bigcup_{t \geq t_0} \mathfrak{C}(t,t_0)$$

The system (1) is function space controllable if, and only if

$$\mathfrak{C}(t_0) = W_2^{(1)}.$$

Equivalently, (1) is function space controllable if for every $\psi \in W_2^{(1)}$ there exists $t_1 \geq t_0$ and a control $u \in \mathbb{P}$ such that the solution $x(t;t_0,0,u)$ of (1) satisfies

$$x_{t_1}(\cdot,t_0,0,u) = \psi.$$

Definition. Consider the linear system

$$\dot{x}(t) = L(t,x_t) \tag{8}$$

where L is defined in (2).

Definition. The trivial solution of (8) is called stable at t_0 if $t_0 \geq 0$ and

(i) there exists $b=b(t_0)>0$ such that if $|\phi| \leq b$ then the solution $x(t_0,\phi)$ of (8) exists for $t \geq t_0$ and $x_t(t_0,\phi)$ is in the domain of definition of L for $t \geq t_0$.

(ii) For every $\varepsilon>0$ there is a $\delta=\delta(t_0,\varepsilon)>0$ such that if $\|\phi\|\leq\delta$ then the solution $x(t_0,\phi)$ of (8) satisfies $\|x_t(t_0,\phi)\| \leq \varepsilon$ for all $t\geq t_0$.

The trivial solution of (8) is called stable if it is stable for every $t_0\geq 0$. It is called uniformly stable if it is stable and the δ above does not depend upon t_0.

Lemma 2.1. (Ref 9 p 91) The trivial solution of (8) is uniformly stable if and only if there is a constant $M>0$ such that

$$|U(t,s)| \leq M, \quad t\geq s\geq 0.$$

Remark. This implies that

$$\|U_t(\cdot s)\| \leq M,$$

if (8) is uniformly stable.

The next result and its Hilbert space analogue are crucial in our investigation.

Lemma 2.2. Let S be a convex set in E^n containing the origin with the property: given any number ε , and any non-zero vector $\eta \in E^n$, there is a vector $y \in S$, such that $\eta^t y \geq \varepsilon$. Then $S = E^n$.

For its proof, see [39, page 7]. We now generalize it in Lemma 2.3.

Lemma 2.3. Let S be a non empty closed convex subset of a Hilbert space H, with the following property: for each $\varepsilon>0$ and each non-zero vector $\eta \in H$, there exists a $y \in S$ such that $(\eta,y)>\varepsilon$. Then S=H. Here $(\cdot,\cdot)$ denotes inner product in H.

Proof. Suppose $S \neq H$, then there is a non-zero vector $\mu \in H$ such that $\mu \notin S$ and such that in

$$\inf_{s \in S} \|s - \mu\| = d > 0$$

for some d. It follows from the discussions on page 49 of [42] that μ and S can be strongly separated by a hyperplane. Consider the point $\lambda \in S$ which is closest to μ. Such a point exists by [42, page 10] corollary 1.4.1. Set

$$v = \frac{\mu - \lambda}{\|\mu - \lambda\|} ;$$

then $v \in H$, and for any $x \in S$

$$(x,v) \leq (\mu,v) - \|\mu - \lambda\|.$$

Hence, (x,v) is bounded above for any $x \in S$, a contradiction, hence S=H.

Remark 2.1. The following two statements are equivalent:

(i) given any number $\varepsilon>0$ and any non-zero vector $\eta \in E^n$, there is a vector of $y \in K \subseteq E^n$ such that $\eta^t y \geq \varepsilon$

(ii) given any number $\varepsilon > 0$

$K \supseteq S_\varepsilon$,

where S_ε is an ε-ball in E^n.

Remark 2.2. The following are equivalent:

(i) given $\varepsilon > 0$ and any non-zero $\eta \in W_2^{(1)}$ there is a vector $y \in K \subseteq W_2^{(1)}$ such that $(\eta,y) \geq \varepsilon$

(ii) given any number $\varepsilon > 0$

$K \supseteq S_\varepsilon$

where S_ε is an ε-ball in $W_2^{(1)}$.

The next definition generalizes the same concept by LaSalle in [39] for non delay systems.

Definition. The system (1) is asymptotically proper in $W_2^{(1)}$ if for each $\varepsilon>0$ and each non-zero vector $\eta \in W_2^{(1)}$ there exists a control $u \in \mathbb{P}$, a time $t_1 \geq t_0$ such that

$$(\eta, \int_{t_0}^{t_1} U_{t_1}(\cdot,s)K(s)u(s)\,ds) \geq \varepsilon. \qquad (9)$$

where $(\cdot,\cdot)$ denotes inner product in $W_2^{(1)}$.

The system is asymptotically proper in E^n if $W_2^{(1)}$ is replaced by E^n and (9) is replaced by

$$\eta^t \int_{t_0}^{t_1} U(t_1,s)K(s,u(s))ds \geq \varepsilon. \tag{10}$$

In section 4, we shall show that function space controllability is equivalent to systems being asymptotically proper when the system (8) is uniformly stable and K satisfies some convexity assumptions.

3. On the Closure and Convexity of Reachable Sets in $W_2^{(1)}$ and E^n.

Theorem 3.1. In (1) assume that

(i) there exists an $N>0$ such that

$$|K(s,u(s))| \leq N\|u\|;\ u \in \mathbb{P},\ s \in I \equiv [t_0,t_1],\ \text{for each } I \subseteq E^1;$$

(ii) the set,

$$\mathbb{K}(t) = \{K(t,u(t)):\ u \in \mathbb{P}\},$$

is convex for each $t \in E^1$;

(iii) the trivial solution of the homogeneous system (8) is uniformly stable.

Then the function space reachable set $\mathcal{C}(t_1,t_0)$ of (1) at time t_1 is closed and convex in $W_2^1((-h,0],E^n)$.

Proof. Because P is compact and I $[t_0,t_1]$ bounded, $\mathbb{P}$ is a bounded subset of $L_2(I,E^m)$. The set $\mathbb{P}$ is also closed in $L_2(I,E^m)$. Indeed, consider a sequence $\{u_k\}$ which converges to an element u in $L_2(I,E^m)$. Then by Theorem 6 p 122 of [27] u_k converges to u in measure, so that a subsequence u_{kn} converges to u almost everywhere (See p 150 of [27] Corollary 13). Since P is closed in E^m, u(t) is an element of P a.e in I. Since u differs by a null function from an element of $\mathbb{P}$, $u \in \mathbb{P}$.

Because P is assumed convex so is $\mathbb{P}$, so that by Theorem 13 p 422 of [27] $\mathbb{P}$ is weakly-closed in $L_2(I,E^m)$. Since $L_2(I,E^m)$ is reflexive we have from Corellary 8 p 425 of [27] that $\mathbb{P}$ is weakly-compact in $L_2(I,E^m)$.

Let $T(u)(\cdot) = \int_{t_0}^{t_1} U_{t_1}(\cdot,s)K(s,u(s))ds$,

be a mapping of $L_2(I,E^m)$ into $W_2^{(1)}$.

Observe that

$$T(\mathbb{P}) = \mathcal{C}(t_1,t_0).$$

Since U is linear and $\mathbb{K}(t)$ convex, $\mathcal{C}(t_1,t_0)$ is convex. We now prove that T is continuous, and hence weakly continuous (see [27], Corollary 5 p 420). Indeed

$$\|T(u)\| \leq \int_{t_0}^{t_1} \|U_{t_1}(\cdot,s)K(s,u(s))\|ds$$

$$\leq \int_{t_0}^{t_1} \|U_{t_1}(\cdot,s)\| \quad |K(s,u(s))|ds.$$

Because (8) is uniformly stable we have by Lemma 2.1 that

$$\|U_{t_1}(\cdot,s)\| \leq M,$$

for some M>0 and all $s \in [t_0,t_1]$. Hence by condition (i) of Theorem 3.1

$$\|T(u)\| \leq NM \int_{t_0}^{t_1} |u(s)|ds.$$

By Hölder inequality,

$$\|T(u)\| \leq NM \sqrt{t_1 - t_0}\|u\|_2 \text{ , so}$$

that T is continuous. Because T is weakly continuous, and $\mathbb{P}$ weakly compact in $W_2^{(1)}$, and so weakly closed.

Since $\mathcal{C}(t_1,t_0)$ is both convex and weakly closed, it follows from Theorem 13 p 422 of [27] that $\mathcal{C}(t_1,t_0)$ is closed in $W_2^{(1)}$. This completes the proof.

Remark. If $K(t,0) = 0$, $0 \in P$ then $0 \in \mathcal{C}(t,t_0)$ for each $t \geq t_0$ and $\mathcal{C}(t_0)$ is nonempty and convex.

Proposition 3.1. In (1) assume that $K(t,0) = 0$, $0 \in P$ then the Euclidean reachable set $\mathbb{R}(t,t_0)$ of (1) at time t is nonempty, and convex. Also, the Euclidean reachable set $\mathbb{R}(t_0)$ of (1) is nonempty and convex.

Proof. Because $0 \in P$ and $K(t,0)=0$ $0 \in \mathbb{R}(t,t_0)$ for each $t \geq t_0$. Since also $\mathbb{R}(t,t_0) \leq R(t_2,t_0)$ $t_0 \leq t_1 \leq t_2$, $0 \in \mathbb{R}(t_0)$. Recall that

$$\mathbb{R}(t,t_0) = \{ \int_{t_0} U(t,s)K(s,u(s))ds: u \in \mathbb{P}\}$$

$$= \int_{t_0}^{t} U(t,s)q(s)ds: q(s) \text{ measurable, } q(s) \in K(s,P)\},$$

so that the convexity of $\mathbb{R}(t,t_0)$ follows from a well-known theorem (see [43] Theorem 3 or [44, Theorem 1]). Since

$$\mathbb{R}(t_0) = \bigcup_{t \geq t_0} \mathbb{R}(t,t_0), \quad 0 \in \mathbb{R}(t,t_0)$$

for each t and $\mathbb{R}(t,t_0) \subseteq \mathbb{R}(t_1,t_0)$ for $t_1 \leq t_2 \mathbb{R}(t_0)$ is also convex. This completes the proof.

4. Controllability.

This section contains the basic results of this chapter from which other results are deduced in the next sections.

Theorem 4.1. In (1) assume that:

(i) there exists an $N>0$ such that

$$|K(t,u(t))| \leq N\|u\|, \quad u \in \mathbb{P}, \quad t \in I, \text{ for each } I \equiv [t_0,t_1];$$

(ii) the set

$$\mathbb{K}(t) = \{K(t,u(t)): u \in \mathbb{P}\}$$

Is convex for each $t \in E^1$;

(iii) the trivial solution of the homogeneous system (8) is uniformly stable.

Then (1) is function space controllable if, and only if it is asymptotically proper in $W_2^{(1)}$.

Proof. The conditions of the theorem yield that the function space reachable set $\mathcal{C}(t_1,t_0)$ is a closed and convex subset of $W_2^{(1)}$. (See Theorem 3.1), for each $t_1 \geq t_0$. Assume that (1) is asymptotically proper in $W_2^{(1)}$. Then for each $\varepsilon>0$ and each non-zero vector $\eta \in W_2^{(1)}$ there exists a control $u \in \mathbb{P}$, a time $t_1 \geq t_0$ such that

$$(\eta, \int_{t_0}^{t_1} U_{t_1}(\cdot,s)K(s,u(s))ds) \geq \varepsilon.$$

Since $y \in \mathcal{C}(t_0)$ is the same as there exists a $t_1 \geq t_0$ a $u \in \mathbb{P}$ such that

$$y = \int_{t_0}^{t_1} U_{t_1}(\cdot,s)K(s,u(s))ds,$$

our assumption that (1) be asymptotically proper is equivalent to: for each $\varepsilon>0$ and each $\eta \in W_2^{(1)}$, there exists a $t_1 \geq t_0$ and a $y \in C(t_1,t_0)$ such that

$$(\eta,y) \geq \varepsilon.$$

It follows from Lemma 2.3 that there exists a $t_1 \geq t_0$ such that

$$W_2^{(1)} = \mathcal{C}(t_1,t_0) ;$$

that is, $W_2^{(1)} = \mathcal{C}(t_0)$.

For necessity, let $\varepsilon>0$, $\eta \in W_2^{(1)}$, and choose $\phi_1 \in W_2^{(1)}$ to satisfy

$$(\eta,\phi_1) \geq \varepsilon.$$

Let $t_1 \geq t_0$, $u \in \mathbb{P}$ be such that

$$x_{t_1}(\cdot,t_0,0,u) = \phi_1.$$

then

$$\int_{t_0}^{t_1} U_{t_1}(\cdot,s)K(s,u(s))ds = \phi_1;$$

and

$$(\eta, \int_{t_0}^{t_1} U_t(\cdot,s)K(s,u(s))ds) \geq \varepsilon.$$

The proof is complete.

Corollary 4.1. Consider the system

$$\dot{x}(t) = L(t,x_t) + B(t)u(t), \quad u \in \mathbb{P} \tag{11}$$

where B is continuous, and L is given by (2). Suppose the trivial solution of (8) is uniformly stable. Then (11) is function space controllable if, and only if (11) is asymptotically proper in $W_2^{(1)}$.

Proof. Since P is convex,

$$\mathbb{K}(t) = \{B(t)P\} \text{ is convex.}$$

Condition (i) is also satisfied: take N to be the uniform bound of B(t) on each compact interval I. The corollary now follows from Theorem 4.1.

Theorem 4.2. In (1) assume that $K(t,0) = 0$, and $0 \in P$. Then (1) is Euclidean controllable if, and only if (1) is asymptotically proper in E^n.

Proof. By the hypothesis and Proposition 3.1 $\mathbb{R}(t,t_0)$, $\mathbb{R}(t_0)$ are both convex and nonempty.

For sufficiency, assume that (1) is asymptotically proper in E^n. Then for each $\varepsilon>0$ and each $\eta \in E^n$, there exists a control $u \in \mathbb{P}$ a time $t_1 \geq t_0$ such that

$$\eta^t \int_{t_0}^{t_1} U(t_1,s)K(s,u(s))ds \geq \varepsilon.$$

that is, for each $\varepsilon>0$ and each $u \in E^n$, there exists a $t_1 \geq t_0$, and $y \in \mathbb{R}(t_1,t_0)$ such that

$$\eta^t y \geq \varepsilon.$$

By Lemma 2.2, we have that there exists a $t_1 \geq t_0$, such that

$$E^n = \mathbb{R}(t_1,t_0);$$

that is,

$$E^n = \bigcup_{t\geq t_0} \mathbb{R}(t,t_0) = \mathbb{R}(t_0).$$

For necessity, let $\varepsilon>0$, $\eta \in E^n$ and choose $x_1 \in E^n$ to satisfy

$$\eta^t(x_1) \geq \varepsilon.$$

Let $t_1 \geq t_0$, $u \in \mathbb{P}$ be selected such that $x(t_1;t_0,0,u) = x_1$.

Then

$$\int_{t_0}^{t_1} U(t_1,s)K(s,u(s))ds = x_1,$$

and

$$\eta^t \int_{t_0}^{t_1} U(t_1,s)u(s,u(s))ds \geq \varepsilon,$$

that is (1) is asymptotically proper in E^n. This completes the proof.

5. Perturbed System.

Let

$$\dot{x}(t) = L(t,x_t) + g(t,u), \tag{12}$$

be a perturbation of the system (1):

$$\dot{x}(t) = L(t,x_t) + K(t,u). \tag{13}$$

Suppose g is continuous in t and u. If our system represents a physical process that involves approximated parameters, the next results give conditions under which the system can be assumed to be controllable.

Theorem 5.1. In (12) and (13) assume that g satisfies the following conditions on K:

(i) there exists an $N>0$ such that

$$|K(t,u(t))| \leq N\|u\|, \quad u \in \mathbb{P}, \quad t \in I,$$

for each $I \equiv [t_0,t_1]$;

(ii) the set

$$\mathbb{K}(t) = \{K(t,u(t)): u \in \mathbb{P}\}$$

is convex for each $t \in I$;

(iii) Suppose the trivial solution of the homogeneous system (8) is uniformly stable;

(iv) Let h be a function defined by

$$h(t) = \sup_{u \in \mathbb{P}} \|U_{t_1}(\cdot,t)\| |g(t,u)-K(t,u)|, \quad t_1 \geq t \geq t_0$$

and assume that h is integrable on $[t_0 \infty)$.

Then (13) is function space controllable if, and only if (12) is function space controllable.

Proof. Define

$$|h| = \int_{t_0}^{\infty} h(s)ds,$$

and let $\epsilon>0$ and $\eta \in W_2^{(1)}$ be given. If (12) is function space controllable, then there exists a $u \in \mathbb{P}$ and a $t \geq t_0$ such that

$$(\eta, \int_{t_0}^{t} U_t(\cdot,s)g(s,u(s))ds) \geq \epsilon + \|h\||h|.$$

Hence,

$$(\eta, \int_{t_0}^{t} U_t(\cdot,s)K(s,u(s))ds)$$

$$= (\eta, \int_{t_0}^{t} U_t(\cdot,s)g(s,u(s))ds)$$

$$- ((\eta, \int_{t_0}^{t} U_t(\cdot,s)g(s,u(s))ds) - (\eta, \int_{t_0}^{t} U_t(\cdot,s)K(s,u(s))ds))$$

$$\geq (\eta, \int_{t_0}^{t} U_t(\cdot,s)g(s,u(s))ds)$$

$$- |(\eta, \int_{t_0}^{t} U_t(\cdot,s)[gs,u(s)) - K(s,u(s))]ds)|,$$

$$\geq \varepsilon + \|\eta\|\|h\| - \|\eta\|\|h\| = \varepsilon .$$

It follows from Theorem 4.1 that (13) is function space controllable. We can prove the converse in the same way. This completes the proof.

Theorem 5.2. In (12) and (13) assume that $K(t,0) = g(t,0) = 0$ and that $0 \in P$. Let h be a function defined by

$$h(t,s) = \sup_{u \in \mathbb{P}} |U(t,s)||g(s,u) - K(s,u)| \quad t \geq s \geq t_0$$

and assume that h is integrable in $s \in [t_0,\infty)$ for each $t \geq s$ then (12) is Euclidean controllable if, and only if (13) is Euclidean controllable.

Proof. Define

$$|h| = \int_{t_0}^{\infty} h(\infty,s)ds,$$

and let $\varepsilon>0$ $\eta \in E^n$ be given. Suppose (12) is Euclidean controllable then there exists a $u \in \mathbb{P}$ and a $t \geq t_0$ such that

$$\eta^t \int_{t_0}^{t} U(t,s)g(s,u(s))ds \geq \varepsilon + |\eta||h| .$$

Proceed as in the proof of Theorem 5.1 to complete the proof.

6. Euclidean Controllability of Perturbed Systems

In this section we examine the Euclidean controllability of the nonlinear delay system

$$\dot{x}(t) = L(t,x_t) + K(t,u) + g(t,x_t,u), \quad t \geq t_0, \tag{14}$$
$$x(t) \equiv 0, \quad \forall\, t \in [t_0-h,t_0],$$

which is a perturbation of (1). We assume as basic that L and K are as given in the previous sections and that g is continuous in all its arguments and $g(t,\phi,0) = 0$, for all $t \in E^1$, $\phi \in W_2^{(1)}$. We assume that for each $t \geq s$ the function $U(t,s)g(s,\phi,u)$ is integrably bounded; that is, there exists a function m(S) which is integrable in $[t_0\infty)$ such that $|U(t,s)g(s,\phi,u)| \leq m(s)$, for all $t \geq s$ $\phi \in W_2^{(1)}$, $u \in \mathbb{P}$.

For systems without delay, this problem was investigated in [45] We now utilize the necessary and sufficient condition in Section 4 and the same approach as in [45] to deduce our result. For a similar investigation where the controls are assumed continuous and unrestrained see the recent paper by Dauer in [46] for a special case of (14).

Theorem 6.1. Assume that $g(t,\phi,u)$ in (14) satisfies a local Lipschitz condition in $\phi \in C$ and that

(i) $U(t,s)g(s,\phi,u)$ is integrably bounded ,

(ii) the set

$\{K(t,u) + g(t,\phi,u): u \in \mathbb{P}\}$,

is convex for all $t \geq t_0$, $\phi \in C$;

(iii) $K(t,0) = g(t,\phi,0) = 0$, $0 \in \mathbb{P}$

Then (14) is Euclidean controllable if, and only if (1) is Euclidean controllable.

The next Lemma is needed in our proof of Theorem 6.1.

Lemma 6.1. Assume that (1) is Euclidean controllable and $U(t,s)g(s,\phi,u)$ integerably bounded. Suppose $K(t,0)=0$, $0 \in \mathbb{P}$. Let $x_1 \in E^n$. Then there exists a time $t_1 \geq t_0$ such that: for any $\phi_t \in C$, $t \in [t_0,t_1] \equiv I$, there exists a control $u \in \mathbb{P}$, such that the solution ψ of

$$x(t) = L(t,x_t) + K(t,u) + g(t,\phi_t,u(t)),$$

$$x(t) \equiv 0, \text{ for } t \in [t_0-h,t_0],$$

satisfies $\psi(t_1) = x_1$.

Proof. Because $U(t,s)g(s,\phi_s,u(s))ds$ in integrably bounded there exists an $N>0$ such that

$$\left| \int_{t_0}^{t} U(t,s)g(s,\phi_s,u(s))ds \right| \leq N$$

for all $t \geq t_0$ $\phi_s \in C$, $s \in [t_0, t_1]$ and $u \in \mathbb{P}$.

It follows from Remark 2.1, Lemma 2.2 and Theorem 4.2 that we can choose a time $t_1 \geq t_0$ so large that if $\varepsilon = 2N+r$, where $r \geq |x_1|$, then

$$\mathbb{R}(t_1,t_0) = \{ \int_{t_0}^{t_1} U(t_1,s)K(s,u(s))ds: \quad u \in \mathbb{P} \} \supseteq S_\varepsilon.$$

It follows from the convexity of $\mathbb{R}(t_1,t_0)$ that

$$x_1 \in \{ \int_{t_0}^{t_1} U(t_1,s)K(s,u(s))ds + g(s,\phi_s,u(s))ds: \quad u \in \mathbb{P} \},$$

for every $\phi_t \in C$, $t \in I$. Hence for every $\phi_t \in C$, $t \in I$ there exists a control $u \in \mathbb{P}$ such that

$$x_1 = \int_{t_0}^{t_1} U(t_1,s)K(s,u(s))ds + g(s,\phi_s,u(s))ds,$$

$$= \psi(t_1),$$

where ψ is the indicated solution. This completes the proof.

Proof of Theorem 6.1. Suppose (1) is not Euclidean controllable, then by Theorem 4.2, (1) is not asymptotically proper. Hence, there exists a $\varepsilon > 0$, a vector $\eta \neq 0$, $\eta^t \in E^n$ such that

$$\eta \int_{t_0}^{t_1} U(t,s)K(s,u))ds \leq \varepsilon,$$

for all $t \geq t_0$ and all $u \in \mathbb{P}$. Since $U(t,s)g(s,\phi,u)$ is integrably bounded, there exists an N such that

$$\eta \int_{t_0}^{t} U(t,s)g(s,\phi,u(s))ds \leq N,$$

for all $t \geq t_0$, $u \in \mathbb{P}$, and $\phi \in C$.

Now choose any $x_1 \in E^n$, such that

$$\eta x_1 > \varepsilon + N.$$

If there exists a $u \in \mathbb{P}$ such that the solution $\psi(t) = \psi(t;t_0,0,u)$ of (14) satisfies $\psi(t_1) = x_1$, for some t_1, then

$$x_1 = \int_{t_0}^{t_1} U(t,s)[K(s,u(s)) + g(s,\psi_s,u(s))]ds,$$

and $\eta x_1 \leq \varepsilon + N$, a contradiction of the way x_1 was chosen. Hence, there does not exist such a $u \in \mathbb{P}$, and a t_1 with the indicated property. This implies that (14) is not Euclidean controllable.

The remainder of the proof is analogous (with slight modification) to the proof of Theorem 2.1 pages 259-261 of [45]. We shall only outline it. Assume (14) is Euclidean controllable and fix $x_1 \in E^n$. Let $t_1 \geq t_0$ be given by Lemma 6.1. Then given any $\psi_0 \in C$, there exists a control $u_0 \in \mathbb{P}$ such that the solution ψ of

$$\dot{x}(t) = L(t,x_t) + K(t,u_0(t)) + g(t,\phi_0,u_0(t)) \quad t \geq t_0$$

$$x(t) \quad 0 \quad \forall\, t \in [t_0-h,t_0]$$

satisfies $\psi(t_1) = x_1$. This solution is given by

$$\psi(t) = \int_{t_0}^{t} U(t,s)[K(s,u_0(s)) + g(s,\phi_0,u(s))]ds. \tag{15}$$

Let

$$\forall(\phi_0) = \{\psi: u \in \mathbb{P},\ \psi \text{ given by (15)},\ \psi(t_1) = x_1\}$$

Then $\forall$ is defined on C.

From Lemma 6.1, $\forall(\phi_0)$, $\phi_0 \in C$ is non empty.

Because K and g are continuous and P compact and because $U(t,s)g(s,\phi,u)$ is integrably bounded $\forall(\phi)$ is bounded for each $\phi \in C$. It is also a convex set for each $\phi \in C$, because of condition (ii) of Theorem 6.1. The arguments cited in [45] carry through to show that $\forall$ has a closed graph; that is, if $\{\phi_i\},\{\psi_i\}$ are two sequences of continuous functions such that

$\|\phi_i - \bar{\phi}\| \to 0$, $\|\psi_i - \bar{\psi}\| \to 0$, as $i \to \infty$,

$\psi_i \in \Psi(\phi_i)$, for each i, then

$\bar{\psi} \in \Psi(\bar{\phi})$.

Let $\mathbb{M}$ denote the colsure of the range of Ψ on C, then $\mathbb{M} \subseteq C$, and $\mathbb{M}$ is bounded. Because K and g are continuous on a compact set and therefore bounded, $\mathbb{M}$ is equicontinuous and thus by Ascolis Theorem compact. We have now shown that

Ψ: $\mathbb{M} \to W(\mathbb{M})$, $W(\mathbb{M})$ is family of closed convex (non empty) subsets of $\mathbb{M}$, and that Ψ is an upper semi continuous mapping.

The result now follows as a consequence of the following Lemma.

Lemma 6.2. [46, Theorem 1]

Let C be a locally convex topological linear space and let $\mathbb{M}$ be a compact convex set in C. Let $W(\mathbb{M})$ be the family of all closed convex (non empty) subsets of $\mathbb{M}$. Then for any upper semicontinuous point-to-set transformation Ψ from $\mathbb{M}$ into $W(\mathbb{M})$ there exists a point $\psi \in \mathbb{M}$ such $\psi \in \Psi(\psi)$. Here upper semicontinuity means that

limit $\psi_n = \psi_0$ $\Phi_n \in \Psi(\psi_n)$ and

limit $\Phi_n = \Phi_0$ implies $\Phi_0 \in \Psi(\psi_0)$.

7. Function Space Controllability of Perturbed Systems.

In Ref. 3, it was proved that the system

$$\dot{x}(t) = A(t)x + B(t)x(t-h) + C(t)u(t), \text{ on } I,$$

$$x(t) = \phi(t), \quad \forall\ t \in [t_0-h, t_0],$$

is function space controllable on $I=[t_0,T]$ $T>t_0+h$, if and only if

$$\dot{x}(t) = A(t)x + B(t)x(t-h) + C(t)u(t) + g(u(t)),$$

$$x(t) = \phi(t) \quad \forall\ t \in [t_0-h, t_0],$$

is function space controllable, provided g is bounded. It was also proved that if (16) is function space controllable then so is the system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) + C(t)u(t) + g(t,x(t),x(t-h)$$

provided g is continuous, locally Lipschitzian in x(t), and x(t-h), and bounded. The controls were taken in $L^2([t_0,T],E^n)$ and are not restricted to a compact convex subset of E^n.

Our aim in this section is to show that when the controls are restricted to a compact and convex subset of E^n, with suitable assumptions, integrably bounded perturbations (14) of function space controllable systems (1) are function space controllable. That there are critical differences between controllability with restraints on the control values and that without such restrictions have been pointed out in [45]. The basic assumptions on L, K, and g is Section 6 are maintained. Our method of proof is similar to that of Section 6. Through the solution states are in $W_2^{(1)}$, we shall work in the larger space $C=C([-h,0],E^n)$ when applying Fan Fixed point Theorem of [22].

Theorem 7.1. In (14), assume that $g(t,\phi,u)$ satisfies a local Lipschitz condition in $\phi \in W_2^{(1)}$, and that it is continuous in all its arguments. Suppose

(i) $U_t(\cdot s)g(s,\phi,u)$ is integrally bounded for $t \geq s$ i.e. there exists an N such that

$$\int_{t_0}^{t} U(t+\theta,s)g(s,\phi,u(s))ds \leq N < \infty,$$

for all $\theta \in [-h,0]$;

(ii) $\mathbb{K}(t) = \{K(t,u(t)): u \in \mathbb{P}\}$

is convex for each t;

(iii) $\mathbb{L}(t) = \{K(t,u(t)) + g(t,\phi,u(t)): u \in \mathbb{P}\}$

is convex for each $t \geq t_0, \phi \in W_2^{(1)}$.

(iv) There exists an $M>0$ such that

$$|K(t,u)| \leq M\|u\|$$

for all $u \in \mathbb{P}$, $t \in I \equiv [t_0,t_1]$ and each I.

(v) The trivial solution of (8) is uniformly stable.

Then (14) is function space controllable if and only if (1) is function space controllable.

We need the following Lemma.

Lemma 7.1. Let the assumptions of Theorem 7.1 hold. Let $\phi \in W_2^{(1)}$. Then there exists a time $t_1 \geq t_0$ such that: For any $\psi \in W_2^{(1)}$ there exists a $u \in \mathbb{P}$ such that the solution $x(t;0,u)$ of

$$\dot{x}(t) = L(t,x_t) + K(t,u(t)) + g(t,\psi,u(t)),$$

$$x(t) \equiv 0 \quad \forall\ t \in [t_0-h,t_0]$$

satisfies $x_{t_1} = \phi$

Proof. From the assumptions, there exists an $N>0$ such that

$$\left\| \int_{t_0}^{t} U_t(\cdot,s)g(s,\psi,u(s))ds \right\| \leq N$$

for all $\psi \in W_2^{(1)}$, $t \geq t_0$ and $u \in \mathbb{P}$. Bacause of the controllability assumption, Remark 2.2 and Theorem 4.1 we can choose a $t_1 \geq t_0$, such that if $\epsilon = 2N+r$, where $r \geq \|\phi\|$, then

$$\mathcal{C}(t_1,t_0) = \{\int_{t_0}^{t_1} U_t(\cdot,s)u(s,u(s))ds \quad u \in \mathbb{P}\} \geq S_\epsilon,$$

where S_ϵ is an ϵ-ball in $W_2^{(1)}$.

Since $\mathcal{C}(t_1,t_0)$ is convex by Theorem 3.1 we have

$$\phi \in \mathcal{C}(t_1,t_0) + \mathcal{C}(t_1,t_0),$$

for every $\psi \in W_2^{(1)}$,

where

$$\mathcal{C}(t_1,t_0) = \{\int_{t_0}^{t_1} U_t(\cdot,s)g(s,\psi,u(s))ds : u \in \mathbb{P}\}.$$

Hence, for every $\phi \in W_2^{(1)}$ there exists a $u \in \mathbb{P}$ such that

$$\phi = \int_{t_0}^{t_1} U_t(\cdot,s)[h(s,u(s)) + g(s,\psi,u(s))]ds$$

$$= x_{t_1},$$

where $x(t,t_0,0,u)$ is a solution of the equation in Lemma 7.1.

Proof of Theorem 7.1. Suppose (1) is not function space controllable, then by Theorem 4.1, (1) is not asymptotically proper: there exists a $\varepsilon>0$, $\eta \in W_2^{(1)}$ η such that

$$(\eta, \int_{t_0}^{t} U_t(\cdot,s)K(s,u(s))ds) \leq \varepsilon$$

for all $t \geq t_0$, $u \in \mathbb{P}$.

By assumption, $U_t(\cdot,s)g(s,\phi,u)$ is integrably bounded, so that there exists an $N>$ such that

$$|(\eta, \int_{t_0}^{t} U_t(\cdot,s)g(s,\psi,u(s))ds)| \leq N\|\eta\|$$

for all $t \geq t_0$, $u \in \mathbb{P}$, $\psi \in W_2^{(1)}$. Now choose $\phi \in W_2^{(1)}$ such that

$$(\eta,\phi) > \varepsilon + N\|\eta\|.$$

If there exists some $u \in \mathbb{P}$ such that the solution $x(t;t_0,0,u)$ of (14) satisfies $x_{t_1} = \phi$ at some t_1, then

$$\phi = \int_{t_0}^{t_1} U_{t_1}(\cdot,s)[K(s,u(s)) + g(s,x_s,u(s))]ds$$

and

$$(\eta,\phi) \leq \varepsilon + N\|\eta\|,$$

which contradicts the choice of ϕ. Hence (14) is not function space controllable.

Conversely, assume (14) is function space controllable and fix $\phi \in W_2^{(1)}$. Let $t_1 \geq t_0$ be given by Lemma 7.1. Then given any $\psi_0 \in W_2^{(1)}$, there exists $u_0 \in \mathbb{P}$, such that the solution $x(t;t_0,0,u)$ of

$$\dot{x}(t) = L(t,x_t) + K(t,u_0(t)) + g(t,\psi_0,u_0(t)),\ t \geq t_0,$$

$$x(t) \equiv 0, \quad \forall\, t \in [t_0-h, t_0],$$

satisfies $x_{t_1} = \phi$. This solution is given by

$$x_t = \int_{t_0}^{t} U_t(\cdot,s)[K(s,u_0(s)) + g(s,\psi_0,u_0(s))]ds \tag{17}$$

Note that ψ_0, $x_t \in W_2^{(1)} \subseteq C$. It is convenient to work in the space C when we apply the Fan-Fixed point theorem.

Let

$$T(\psi_0) = \{x_t \in C: u \in \mathbb{P},\ x \text{ satisfies (17)},\ x_{t_1} = \phi\}$$

Just as in the proof of Theorem 6.1. $T(\psi_0)$ is nonempty, convex and bounded for each $\psi_0 \in C$. T also has a closed graph. The proof of this last assertion follows that given in [45] p 259-260. If $\mathbb{M}$ denotes the closure of the range of T on C then $\mathbb{M} \subseteq C$, and we can prove, just as in [45] that $\mathbb{M}$ is compact. Hence T maps $\mathbb{M}$ into a family of closed convex nonempty subsets of $\mathbb{M}$ and T is upper semicontinuous. The Fan Fixed point theorem, Lemma 6.2, yields a fixed point $\psi \in T(\psi)$ which is the desired solution.

VII. Conclusion

In chapter II, we generalize the Liapunov-Yoshizawa techniques to give necessary and sufficient conditions for uniform boundedness and uniform ultimate boundedness of a rather general class of nonlinear differential equations of neutral type. This is then applied to several linear and nonlinear systems of equations including the generalized Lienard equation of neutral type. The calculations seem less cumbersome than is the case when Liapunov-Razamikhin techniques are adopted as suggested in [3]. Only explicit Liapunov-Yoshizawa functionals are utilized in the applications.

In chapter III, we apply the converse theorems of [15], [1] and chapter II, to investigate the boundedness of ordinary and hereditary systems of Lurie type. When the uncontrolled systems are assumed to be uniform bounded and uniform ultimate bounded and when $\int_0^\sigma f(s)ds \to \infty$ as $|\sigma| \to \infty$, $f(0) = 0$, $\sigma f(\sigma) > 0$ if $\sigma \neq 0$, conditions are obtained for the uniform boundedness of nonlinear ordinary differential systems and hereditary systems of Lurie type described by the equations

$$\dot{x}(t) = A(t,x) + bf(\sigma),$$

$$\dot{\sigma}(t) = B(t,x) - rf(\sigma);$$

$$\dot{x}(t) = F(t,x_t) + bf(\sigma),$$

$$\dot{\sigma}(t) = E(t,x_t) - rf(\sigma);$$

(1) $$\frac{d}{dt}(D(t)x_t) = F(t,x_t) + bf(\sigma)$$

(2) $$\dot{\sigma}(t) = G(t,D(t)x_t) - rf(\sigma).$$

In chapter IV, the problem of Lurie for system described by (1) and (2) is posed. Sufficient conditions are obtained for absolute

stability for the controlled system if it is assumed that the uncontrolled plant equation

(3) $$\frac{d}{dt}(D(t)x_t) = F(t,x_t)$$

is uniformly asymptotically stable. Both the direct and indirect control cases are treated.

In chapter V, we use Fan fixed pointed theorem to prove the existence of a solution of the neutral functional inclusion

$$\frac{d}{dt} D(t,x_t) \in R(t,x_t)$$

which satisfies the two point boundary values

$$x_{t_0} = \phi_0, \quad x_{t_1} = \phi_1$$

where $\phi_0,\phi_1 \in C = C([-h,0],E^n)$. We then apply this existence result to present sufficient conditions on f, D and Ω which imply exact controllability between two functions in C for the system

$$\frac{d}{dt} D(t,x_t) = f(t,x_t u), \quad u(t) \in \Omega(t,x_t).$$

Using a geometric growth condition in chapter VI, we characterize both the function space and Euclidean controllability of a nonlinear delay system which has a compact and convex control set. This extends analogous results for ordinary differential equations and yields conditions under which perturbed nonlinear delay controllable systems are controllable.

The treatment in chapter VI on controllability of systems with limited control power is not quite complete. Consider the system

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) + B_0u(t)$$

for example. The treatment in chapter VI does not give easily verifiable conditions for controllability. The following problem is suggested. Introduce the notion of a proper control system. This concept should be equivalent to controllability for delay systems with unlimited control power. Prove that if the uncontrolled system $\dot{x} = L(t,x_t)$ is uniformly asymptotically stable and the control equation

$$\dot{x}(t) = L(t,x_t) + B(t)u(t)$$

is proper then the control system is controllable.

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16 Abstract

By generalizing the Liapunov-Yoshizawa techniques we give necessary and sufficient conditions for uniform boundedness and uniform ultimate boundedness of a rather general class of nonlinear differential equations of neutral type. Among the applications treated by our methods are the Lienard equation of neutral type and hereditary systems of Lurie type. The absolute stability of this later equation is also investigated.

We next apply a certain existence result of a solution of a neutral functional differential inclusion with two point boundary values to study the exact function space controllability of a nonlinear neutral functional differential control system. We then use a geometric growth condition to characterize both the function space and Euclidean controllability of another nonlinear delay system which has a compact and convex control set. This yields conditions under which perturbed nonlinear delay controllable systems are controllable.

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